

Regular Polygonal Complexes in Space, I *

Daniel Pellicer

York University

Toronto, Ontario, Canada M3J 1P3

and

Egon Schulte[†]

Northeastern University

Boston, MA 02115, USA

Abstract

A polygonal complex in euclidean 3-space \mathbb{E}^3 is a discrete polyhedron-like structure with finite or infinite polygons as faces and finite graphs as vertex-figures, such that a fixed number $r \geq 2$ of faces surround each edge. It is said to be regular if its symmetry group is transitive on the flags. The present paper and its successor describe a complete classification of regular polygonal complexes in \mathbb{E}^3 . In particular, the present paper establishes basic structure results for the symmetry groups, discusses geometric and algebraic aspects of operations on their generators, characterizes the complexes with face mirrors as the 2-skeletons of the regular 4-aperoptopes in \mathbb{E}^3 , and fully enumerates the simply flag-transitive complexes with mirror vector $(1, 2)$. The second paper will complete the enumeration.

Key words. regular polyhedron, regular polytope, abstract polytope, complex.

MSC 2000. Primary: 51M20. Secondary: 52B15.

1 Introduction

The study of highly symmetric polyhedra-like structures in ordinary euclidean 3-space \mathbb{E}^3 has a long and fascinating history tracing back to the early days of geometry. With the passage of time, various notions of polyhedra have attracted attention and have brought

*Version of June 3, 2009

[†]Supported by NSA-grant H98230-07-1-0005

to light new exciting classes of regular polyhedra including well-known objects such as the Platonic solids, Kepler-Poinsot polyhedra, Petrie-Coxeter polyhedra, or the more recently discovered Grünbaum-Dress polyhedra (see [5, 4, 9, 10, 13]).

The radically new *skeletal* approach to polyhedra pioneered in [13] is essentially graph-theoretical and has had an enormous impact on the field. Since then, there has been a lot of activity in this area, beginning with the full enumeration of the 48 “new” regular polyhedra in \mathbb{E}^3 by Grünbaum [13] and Dress [9, 10] (with a simpler approach described in the joint works [20, 21] of the second author with McMullen), moving to the full enumeration of chiral polyhedra in \mathbb{E}^3 in [30, 31] (see also [27]), then continuing with corresponding enumerations of regular polyhedra, polytopes, or apeirotopes (infinite polytopes) in higher-dimensional euclidean spaces by McMullen [17, 18, 19] (see also [1, 3]). For a survey, see [22].

The present paper and its successor [26] describe a complete classification of regular polygonal complexes in euclidean space \mathbb{E}^3 . Polygonal complexes are discrete polyhedra-like structures made up from convex or non-convex, planar or skew, finite or infinite (helical or zig-zag) polygonal faces, always with finite graphs as vertex-figures, such that every edge lies in at least two, but generally $r \geq 2$ faces, with r not depending on the edge. As combinatorial objects they are incidence complexes of rank 3 with polygons as 2-faces (see [7, 29]). Polyhedra are precisely the polygonal complexes with $r = 2$. A polygonal complex is said to be *regular* if its full euclidean symmetry group is transitive on the flags. The two papers are part of a continuing program to classify discrete polyhedra-like structures by transitivity properties of their symmetry group. Characteristic of this program is the interplay of the abstract, purely combinatorial, aspect and the geometric one of realizations of complexes and their symmetries.

The paper is organized as follows. In Sections 2 and 3, respectively, we begin with the notion of a regular polygonal complex \mathcal{K} and establish basic structure results for its symmetry group $G(\mathcal{K})$ in terms of generators for distinguished subgroups. It is found that either \mathcal{K} is simply flag-transitive, meaning that its *full* symmetry group $G(\mathcal{K})$ has trivial flag stabilizers, or that \mathcal{K} has face-mirrors, meaning that \mathcal{K} has planar faces lying in reflection mirrors and that $G(\mathcal{K})$ has flag stabilizers of order 2 (containing the reflection in the plane spanned by the face in a flag). Then in Section 4 we characterize the regular complexes with face mirrors as the 2-skeletons of the regular 4-apeirotopes in \mathbb{E}^3 . There are exactly four such complexes, up to similarity. In Section 5 we discuss operations on the generators of $G(\mathcal{K})$ that allow us to construct new complexes from old and that help reduce the number of cases to be considered in the classification. Finally, in Sections 6 and 7, respectively, we enumerate the simply flag-transitive regular polygonal complexes with finite or infinite faces and mirror vector $(1, 2)$. Apart from polyhedra, there are exactly eight polygonal complexes of this kind, up to similarity.

In the subsequent paper [26] we investigate other possible mirror vectors and employ the operations to complete the classification.

2 Polygonal complexes

Geometric realizations of abstract polytopes or complexes have attracted a lot of attention. One specific approach is to start with an abstract object and then study ways of realizing its combinatorial structure in a more explicit geometric setting (for example, see [21, Ch.5] and [23]). In this paper, however, we take a more direct geometric approach and explicitly define the objects as polyhedra-like structures in an ambient space. In fact, throughout, this space will be euclidean 3-space \mathbb{E}^3 .

Informally, a polygonal complex will consist of a family of vertices, edges and (finite or infinite) polygons, all fitting together in a way characteristic for geometric polyhedra or polyhedral complexes.

We say that a *finite polygon*, or simply *n-gon*, (v_1, v_2, \dots, v_n) in euclidean 3-space \mathbb{E}^3 is a figure formed by distinct points v_1, \dots, v_n , together with the line segments (v_i, v_{i+1}) , for $i = 1, \dots, n-1$, and (v_n, v_1) . Similarly, an *infinite polygon* consists of a sequence of distinct points $(\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots)$ and of line segments (v_i, v_{i+1}) for each i , such that each compact subset of \mathbb{E}^3 meets only finitely many line segments. In either case we refer to the points as *vertices* and to the line segments as *edges* of the polygon.

Definition 2.1. A polygonal complex, or simply complex, \mathcal{K} in \mathbb{E}^3 consists of a set \mathcal{V} of points, called vertices, a set \mathcal{E} of line segments, called edges, and a set \mathcal{F} of polygons, called faces, such that the following properties are satisfied.

- a) The graph defined by \mathcal{V} and \mathcal{E} , called the edge graph of \mathcal{K} , is connected.
- b) The vertex-figure of \mathcal{K} at each vertex of \mathcal{K} is connected. By the vertex-figure of \mathcal{K} at a vertex v we mean the graph, possibly with multiple edges, whose vertices are the neighbors of v in the edge graph of \mathcal{K} and whose edges are the line segments (u, w) , where (u, v) and (v, w) are edges of a common face of \mathcal{K} .
- c) Each edge of \mathcal{K} is contained in exactly r faces of \mathcal{K} , for a fixed number $r \geq 2$.
- d) \mathcal{K} is discrete, meaning that each compact subset of \mathbb{E}^3 meets only finitely many faces of \mathcal{K} .

We call a polygonal complex a (geometric) *polyhedron* if $r = 2$. For general properties of finite or infinite polyhedra in \mathbb{E}^3 we refer to [21, Ch.7E] and [13, 20]. Note that the underlying combinatorial “complex” given by the vertices, edges and faces of a polygonal complex \mathcal{K} , ordered by inclusion, is an incidence complex of rank 3 in the sense of [7, 29]. (When dealing with incidence complexes of rank 3, we are suppressing their improper faces: the unique minimum face of rank -1 and the unique maximum face of rank 3.) In the next section we shall require some basic structure results about the automorphism groups of incidence complexes obtained in [29].

Note that the discreteness assumption in Definition 2.1(d) implies that the vertex-figure at every vertex is finite, and that, for complexes with regular polygons as faces (with “regular” to be defined below), each compact subset of \mathbb{E}^3 meets only finitely many

vertices and edges (in fact, under the regularity assumption on the faces, this could have served as an alternative definition for discreteness).

The (geometric) *symmetry group* $G = G(\mathcal{K})$ of a polygonal complex \mathcal{K} consists of all isometries of the affine hull of \mathcal{K} that map \mathcal{K} to itself. Note that this affine hull is \mathbb{E}^3 , except when \mathcal{K} is planar. The symmetry group $G(\mathcal{K})$ can be viewed as a subgroup of the (combinatorial) *automorphism group* $\Gamma(\mathcal{K})$ of \mathcal{K} , which consists of the combinatorial automorphisms of the underlying combinatorial (incidence) complex, that is, of the incidence preserving bijections of the set of vertices, edges and faces of \mathcal{K} . Throughout we use the term *group of \mathcal{K}* to mean the (full) symmetry group $G(\mathcal{K})$ of \mathcal{K} .

A polygonal complex \mathcal{K} is called (geometrically) *regular* if its symmetry group $G(\mathcal{K})$ is transitive on the flags of \mathcal{K} , or *combinatorially regular* if its automorphism group $\Gamma(\mathcal{K})$ is transitive on the flags of \mathcal{K} . Recall here that a *flag* of \mathcal{K} is a 3-element set consisting of a vertex, an edge and a face of \mathcal{K} , all mutually incident. Two flags of \mathcal{K} are said to be *i-adjacent* if they differ precisely in their element of rank i , that is, their vertex, edge, or face if $i = 0, 1$ or 2 , respectively. Every flag of a polygonal complex \mathcal{K} is *i-adjacent* to exactly one other flag if $i = 0$ or 1 , or exactly $r - 1$ other flags if $i = 2$.

The (symmetry) group $G(\mathcal{K})$ of a regular polygonal complex \mathcal{K} is transitive, separately, on the vertices, edges, and faces of \mathcal{K} . In particular, the faces are necessarily regular polygons, either finite, planar (convex or star-) polygons or non-planar (*skew*) polygons, or infinite, planar zigzags or helical polygons (see [6, Ch.1] or [13, 20]). Note that a regular complex \mathcal{K} cannot have faces which are linear apeirogons; in fact, by the connectedness assumptions on \mathcal{K} , this would force \mathcal{K} to “collapse” onto a single apeirogon.

Notice also that we explicitly allow the vertex-figures of \mathcal{K} to have multiple edges, to account for the possibility that two adjacent edges of a face are adjacent edges of more than one face. On the other hand, if \mathcal{K} is regular, then all edges of vertex-figures have the same multiplicity and all are single or double edges. In fact, if two adjacent edges of a face are adjacent edges of another face, then the symmetry of \mathcal{K} that fixes these two edges and maps the first face to the second face must necessarily be the reflection in the plane spanned by these two edges. This also shows that regular complexes with planar faces have vertex-figures with single edges; that is, they are simple graphs.

In practice we can think of the vertex-figure of \mathcal{K} at a vertex v as the graph whose vertices are given by the neighboring vertices of v in the edge graph of \mathcal{K} and whose edges are represented by straight line segments, with double edges (if any) lying on top of each other.

A priori there seems to be no tractable duality theory for polygonal complexes in space. At the combinatorial level there are no problems: each incidence complex has a dual which is also incidence complex, of the same rank as the original (see [7, 29]). The dual of an incidence complex associated with a polygonal complex in space has r “vertices” on an “edge”, such that the “vertex-figures” are isomorphic to polygons. Configurations of points and lines in space (with r points on every line) come to mind as candidates for duals, but there seems to be no obvious way of relating configurations and polygonal complexes

geometrically. Moreover, as a reminder of the limitations of the duality concept, even the class of geometric polyhedra (that is when $r = 2$) is not closed under geometric duality (see [21, 7E]): there are examples of regular polyhedra which do not have a geometric dual which is also a regular polyhedron (any helix-faced polyhedron gives an example).

The underlying edge graphs of regular complexes are examples of highly symmetric “nets” as studied in crystal chemistry (for example, see [8, 24, 25], and note in particular that [24] describes the nets of many regular polyhedra using the notation and nomenclature of [21, Ch. 7E]). Nets are infinite, periodic geometric graphs in space that represent crystal structures, with vertices corresponding to atoms and edges to bonds. From a chemist’s perspective, the arrangement of the atoms in space is a central piece of information about a chemical compound. The famous *diamond net* is the underlying net of the diamond form of carbon and of several other compounds. In the diamond net, each vertex v is joined to exactly four neighboring vertices such that any two neighbors, along with v , are vertices of exactly two hexagonal “rings” (fundamental circuits) of the net; in [25, p. 299], the local configuration is described by the “long Schläfli symbol” $6_2 \cdot 6_2 \cdot 6_2 \cdot 6_2 \cdot 6_2 \cdot 6_2$ (there are six choices for the two neighbors and each contributes 2 rings with 6 vertices). The diamond net with its hexagonal ring structure will occur in our enumeration as the underlying edge-graph of the complex $\mathcal{K}_7(1, 2)$ of (14). Its faces are skew hexagons, six around each edge, such that the vertex-figure is the double edge-graph of the tetrahedron (the six edges of the tetrahedral vertex-figure, each counted twice, contribute the six terms 6_2 in the above symbol).

3 The symmetry group

In this section we establish some basic structure theorems about the (symmetry) group $G(\mathcal{K})$ of a regular (polygonal) complex \mathcal{K} .

Let \mathcal{K} be a regular complex, and let $G := G(\mathcal{K})$ be its group. Let $\Phi := \{F_0, F_1, F_2\}$ be a fixed, *base*, flag of \mathcal{K} , where F_0 is a vertex, F_1 an edge and F_2 a face of \mathcal{K} . We denote the stabilizers in G of $\Phi \setminus \{F_i\}$ and more generally of a subset Ψ of Φ by

$$G_i = G_i(\mathcal{K}) := \{R \in G \mid F_j R = F_j \text{ for } j \neq i\} \quad (i = 0, 1, 2)$$

and

$$G_\Psi := \{R \in G \mid FR = F \text{ for } F \in \Psi\},$$

respectively. Then $G_i = G_{\{F_j, F_k\}}$ with i, j, k distinct, and G_Φ is the stabilizer of the base flag Φ . Moreover,

$$G_\Phi = G_0 \cap G_1 = G_0 \cap G_2 = G_1 \cap G_2.$$

We also set $G_{F_i} := G_{\{F_i\}}$ for $i = 0, 1, 2$; this is the stabilizer of F_i . The stabilizer subgroups satisfy the following properties.

Lemma 3.1. *Let \mathcal{K} be a regular complex with group $G = G(\mathcal{K})$. Then,*

- a) $G = \langle G_0, G_1, G_2 \rangle$;
- b) $G_\Psi = \langle G_j \mid F_j \notin \Psi \rangle$, for all $\Psi \subseteq \Phi$;
- c) $G_{\Psi_1} \cap G_{\Psi_2} = G_{\Psi_1 \cup \Psi_2}$, for all $\Psi_1, \Psi_2 \subseteq \Phi$;
- d) $\langle G_j \mid j \in I \rangle \cap \langle G_j \mid j \in J \rangle = \langle G_j \mid j \in I \cap J \rangle$, for all $I, J \subseteq \{0, 1, 2\}$;
- e) $G_0 \cdot G_2 = G_2 \cdot G_0 = G_{F_1}$.

Proof. These statements for stabilizer subgroups of G are particular instances of similar such statements for flag-transitive subgroups of the automorphism group of a regular incidence complex of rank 3 (or higher) obtained in [29, §2]. The proof of the corresponding statements for the automorphism groups of polyhedra can also be found in [21, pp.33,34].

The proofs in [29] rest on a crucial connectedness property (strong flag-connectedness) of incidence complexes, which in terms of \mathcal{K} can be described as follows. Two flags of \mathcal{K} are said to be *adjacent* (*j-adjacent*) if they differ in a single face (just their *j*-face, respectively). Here the number of flags *j*-adjacent to a given flag is 1 if $j = 1$ or 2, or $r - 1$ if $j = 0$. Now the connectedness assumptions on \mathcal{K} in Definition 2.1(a,b) are equivalent to \mathcal{K} being *strongly flag-connected*, in the sense that, if Υ and Ω are two flags of \mathcal{K} , then they can be joined by a sequence of successively adjacent flags $\Upsilon = \Upsilon_0, \Upsilon_1, \dots, \Upsilon_k = \Omega$, each containing $\Upsilon \cap \Omega$. Thus, if Ψ is a subset of the base flag Φ , then any flag Ω containing Ψ can be joined to Φ by a similar such sequence in which all flags contain Ψ . This key fact is the basis for an inductive argument. In fact, it can be shown by induction on k that Ω is the image of Φ under an element of the subgroup G_Ψ of G . Since the flag stabilizer G_Φ is a subgroup of G_Ψ , this then proves part (b) and, in turn, parts (a), (c) and (d). Finally, part (e) reflects the fact that, for flags containing the base edge F_1 , the two operations of taking 0-adjacent or 2-adjacent “commute”. \square

Call an affine plane in \mathbb{E}^3 a *face mirror* of \mathcal{K} if it contains a face of \mathcal{K} and is the mirror of a reflection in $G(\mathcal{K})$. Regular complexes with face mirrors must have planar faces. Moreover, if any one face of a regular complex \mathcal{K} determines a face mirror, so do all faces, by the face-transitivity of $G(\mathcal{K})$.

Lemma 3.2. *Let \mathcal{K} be a regular complex with base flag Φ and group $G = G(\mathcal{K})$.*

- a) *Then G_Φ has order at most 2.*
- b) *If G_Φ is non-trivial, then \mathcal{K} has planar faces and the one non-trivial element of G_Φ is the reflection in the plane containing the base face F_2 ; in particular, \mathcal{K} has face mirrors.*

Proof. Let $R \in G_\Phi$, that is, $F_i R = F_i$ for $i = 0, 1, 2$. Then R fixes F_0 , the midpoint of F_1 , and either the center of F_2 if F_2 is a finite polygon, or the axis of F_2 if F_2 is an infinite polygon. Recall here that the faces of \mathcal{K} are regular polygons. (The axis of a zigzag polygon is the line through the midpoints of its edges, and the axis of a helical polygon is the “central” line of the “spiral” staircase it forms.)

If the faces of \mathcal{K} are finite, then R fixes three non-collinear points, and hence R must be either trivial or the reflection in the plane spanned by the three points. In particular, the latter forces the faces to be planar.

If the faces are helical, then necessarily R is trivial. In fact, since R keeps the line through F_1 pointwise fixed and leaves the axis of F_2 invariant, this axis must necessarily intersect this line if R is non-trivial. However, the latter cannot occur.

Finally, if the faces are zigzags, then R again fixes the line through F_1 pointwise and leaves the axis of F_2 invariant. Since these two lines are not orthogonal to each other, R must necessarily fix the plane containing F_2 pointwise, and, if not trivial, coincide with the reflection in this plane.

In either case, G_Φ has at most two elements. In particular, G_Φ can only be non-trivial if F_2 is planar and the plane through F_2 is a face mirror. \square

In Section 4 we will explicitly characterize the regular complexes with a non-trivial flag stabilizer G_Φ . Here we determine the structure of their subgroups G_0, G_1 and G_2 .

Lemma 3.3. *Let \mathcal{K} be a regular complex with a non-trivial flag stabilizer G_Φ (of order 2). Then,*

- a) $G_0 \cong C_2 \times C_2 \cong G_1$;
- b) $G_2 \cong D_r$, the dihedral group of order $2r$.

Proof. Recall that the non-trivial element R in G_Φ is the reflection in the plane containing the base face F_2 . First note that the index of G_Φ in G_0 is 2 (any element in G_0 either fixes Φ or moves Φ to the flag that is 0-adjacent to Φ). Suppose that $S \in G_0$ and $F_0 S \neq F_0$. Then $F_0 S^2 = F_0$ and hence $S^2 \in G_\Phi$. On the other hand, S^2 is a proper (orientation preserving) isometry and hence cannot coincide with R . Thus $S^2 = I$, the identity mapping. It follows that every element of G_0 has period 2, so G_0 must be isomorphic to $C_2 \times C_2$. Similarly we also obtain $G_1 \cong C_2 \times C_2$, using F_1 instead of F_0 .

Moreover, since G_2 fixes the line through F_1 pointwise and contains a reflection, namely R , the group G_2 must be dihedral of order $2r$. Note here that G_2 is transitive on the flags that are 2-adjacent to Φ , and that R fixes F_2 . \square

The vertex-figures of a regular complex \mathcal{K} with non-trivial flag-stabilizer are simple graphs on which the corresponding vertex-stabilizer subgroup of $G(\mathcal{K})$ acts flag-transitively but not simply flag-transitively (the reflection in a face mirror of \mathcal{K} acts trivially on a flag of the vertex-figure at a vertex of this face). Recall here that a flag of a graph (possibly with multiple edges) is a 2-element set consisting of a vertex and an edge incident with this vertex.

We call a regular complex \mathcal{K} *simply flag-transitive* if its full symmetry group $G(\mathcal{K})$ is simply transitive on the flags of \mathcal{K} . Note here that our terminology strictly refers to a condition on the full group $G(\mathcal{K})$, not a subgroup. Nevertheless, regular complexes \mathcal{K} that are not simply flag-transitive in this sense, frequently admit proper subgroups of $G(\mathcal{K})$ that do act simply flag-transitively on \mathcal{K} .

Lemma 3.4. *Let \mathcal{K} be a simply flag-transitive regular complex. Then,*

- a) $G_0 = \langle R_0 \rangle$ and $G_1 = \langle R_1 \rangle$, for some point, line or plane reflection R_0 and some line or plane reflection R_1 ;
- b) G_2 is a cyclic or dihedral group of order r (in particular, r is even if G_2 is dihedral);
- c) $G_{F_2} = \langle R_0, R_1 \rangle$, a group isomorphic to a dihedral group (of finite or infinite order) and acting simply transitively on the flags of \mathcal{K} containing F_2 ;
- d) $G_{F_0} = \langle R_1, G_2 \rangle$, a finite group acting simply transitively on the flags of \mathcal{K} containing F_0 .

Proof. Since G_Φ is trivial, the subgroups G_0 and G_1 are of order 2 (recall here that G_Φ has index 2 in G_0 and G_1) and hence are generated by involutions R_0 and R_1 , respectively; the latter must necessarily be reflections in points, lines or planes. Note that R_1 cannot be a point reflection (in the point F_0), since otherwise this would force F_2 to be a linear apeirogon. Now parts (c) and (d) follow directly from Lemma 3.1(b) (with $\Psi = \{F_2\}$ or $\Psi = \{F_0\}$, respectively). Here G_{F_0} is finite, since \mathcal{K} is discrete.

For part (b) note that G_2 keeps the line through F_1 pointwise fixed and hence is necessarily cyclic or dihedral. Since G_2 acts simply transitively on the flags of \mathcal{K} containing F_0 and F_1 , it must have order r . \square

For a simply flag-transitive regular complex \mathcal{K} , the vertex-stabilizer subgroup G_{F_0} of the base vertex F_0 in G acts simply flag-transitively on (the graph that is) the vertex-figure of \mathcal{K} at F_0 . Its order is twice the number of faces of \mathcal{K} containing F_0 (that is, twice the number of edges of the vertex-figure). We call G_{F_0} the *vertex-figure group* of \mathcal{K} at F_0 . Even if the vertex-figure is a simple (geometric) graph, this group may *a priori* not be its full symmetry group in \mathbb{E}^3 . If the vertex-figure has double edges, then any two edges with the same pair of end vertices can be mapped onto each other by an element of G_{F_0} (swapping the two faces of \mathcal{K} that determine these edges). On the other hand, the face stabilizer G_{F_2} always turns out to be the full symmetry group of the face (face mirrors do not occur if \mathcal{K} is simply flag-transitive).

The following theorem settles the enumeration of finite complexes. Recall that there are precisely eighteen finite regular polyhedra in \mathbb{E}^3 , namely the Platonic solids and Kepler-Poinsot polyhedra, and their Petrie-duals (see [21, Ch.7E]).

Theorem 3.5. *The only finite regular complexes in \mathbb{E}^3 are the finite regular polyhedra.*

Proof. First note that every finite subgroup of $\mathcal{O}(3)$ (the orthogonal group of \mathbb{E}^3) leaves a point invariant. Thus the center (centroid of the vertex set) of a finite regular complex \mathcal{K} is invariant under G . Now assume to the contrary that $r \geq 3$. Then G_2 must contain a rotation about the line through F_1 , which, in turn, forces the center of \mathcal{K} to lie on this line. But then all lines through edges of \mathcal{K} must pass through the center of \mathcal{K} , which is impossible. \square

Thus we may restrict ourselves to the enumeration of infinite complexes. For the remainder of this section we will assume that \mathcal{K} is a simply flag-transitive regular complex and that \mathcal{K} is infinite. We continue to explore the structure of G .

First we say more about the stabilizer G_{F_1} of the base edge F_1 . In particular, G_2 has index 2 in G_{F_1} , and

$$G_{F_1} = G_0 G_2 \cong G_2 \rtimes G_0. \quad (1)$$

Note here that $G_0 \cap G_2$ is trivial, since \mathcal{K} is simply flag-transitive. The semi-direct product in (1) is direct if R_0 is a point reflection (in the midpoint of F_1) or a plane reflection (in the perpendicular bisector of F_1).

The group G of \mathcal{K} is an infinite discrete group of isometries of \mathbb{E}^3 . We say that such a group acts (*affinely*) *irreducible* if there is no non-trivial linear subspace L of \mathbb{E}^3 which is invariant in the sense that G permutes the translates of L (and hence also the translates of the orthogonal complement L^\perp). Otherwise, the group is called (*affinely*) *reducible*.

Our next theorem allows us to concentrate on regular complexes with an irreducible group. Note that \mathcal{K} need not be simply flag-transitive for the purpose of this theorem.

Theorem 3.6. *A regular complex in \mathbb{E}^3 with an affinely reducible symmetry group is necessarily a regular polyhedron and hence is planar or blended in the sense of [21, pp.221,222].*

Proof. We need to rule out the possibility that $r \geq 3$. Suppose to the contrary that $r \geq 3$ and that G is reducible. Let L be a plane in \mathbb{E}^3 whose translates are permuted under G (this is the only case we need to consider). In particular, the images of L under G_2 are translates of L . Since G_2 is cyclic or dihedral of order at least 3, we know that its rotation subgroup is generated by a rotation S about the line through F_1 .

If S is of order at least 3, then its axis must necessarily be orthogonal to L . By taking a plane parallel to L if need be, we may assume that L is the perpendicular bisector of F_1 . Then, by the edge-transitivity of G , each perpendicular bisector of an edge of \mathcal{K} is an image of L under G and hence must be parallel to L . However, this forces all edges to be parallel, which is impossible (a complex cannot have linear apeirogons as faces).

Now let S be of order 2. The case when the axis of S is orthogonal to L can be eliminated as before. It remains to consider the case when the axis of S is parallel to L and hence, without loss of generality, is contained in L . Then, by the edge-transitivity of G , every edge of \mathcal{K} is contained in an image of L under G , and by the connectedness of \mathcal{K} , that image, being parallel to L , must necessarily be L itself. Hence \mathcal{K} must lie in a plane. Finally, to exclude this possibility note that a line segment in a plane can be an edge of at most two regular (finite or infinite) polygons of a given shape (consider the interior angles at vertices of the polygon). This forces $r = 2$, contrary to our assumption. \square

It is well known that an irreducible infinite discrete group of isometries in \mathbb{E}^3 is a crystallographic group (that is, it admits a compact fundamental domain). The well-known Bieberbach theorem now tells us that such a group G contains a subgroup $T(G)$

of the group $\mathcal{T}(\mathbb{E}^3)$ of translations of \mathbb{E}^3 , such that the quotient $G/T(G)$ is finite; here we may think of $T(G)$ as a lattice in \mathbb{E}^3 (see [2] and [28, §7.4]). If $R : x \mapsto xR' + t$ is a general element of G , with $R' \in \mathcal{O}(3)$ and $t \in \mathbb{E}^3$ a translation vector (we may thus think of $t \in \mathcal{T}(\mathbb{E}^3)$), then the mappings R' clearly form a subgroup G_* of $\mathcal{O}(3)$, called the *special group* of G . Thus G_* is the image of G under the epimorphism $\mathcal{I}(3) \mapsto \mathcal{O}(3)$ whose kernel is $\mathcal{T}(\mathbb{E}^3)$. (Here $\mathcal{I}(3)$ is the group of all isometries of \mathbb{E}^3 .) In other words,

$$G_* = G\mathcal{T}(\mathbb{E}^3)/\mathcal{T}(\mathbb{E}^3) \cong G/(G \cap \mathcal{T}(\mathbb{E}^3)) = G/T(G),$$

if $T(G)$ is the full translation subgroup of G . In particular, G_* is a finite group.

In essence, the following lemma was proved in [21, p.220] (see also [31, Lemma 3.1]). For the exclusion of rotations of period 6 for groups in \mathbb{E}^3 observe that there is no finite subgroup of $\mathcal{O}(3)$ with two distinct 6-fold axes of rotation, and that hence the presence of 6-fold axes forces reducibility.

Lemma 3.7. *The special group of an irreducible infinite discrete group of isometries in \mathbb{E}^2 or \mathbb{E}^3 does not contain rotations of periods other than 2, 3, 4 or 6. Moreover, a rotation of period 6 can only occur in the planar case.*

Lemma 3.7 immediately limits the possible values of r . In fact, we must have $r = 2, 3, 4, 6$ or 8 . Similarly, the faces, if finite, must be p -gons with $p = 3, 4, 6$ or 8 . We later eliminate some of these possibilities.

The translation subgroup of the symmetry group (or any flag-transitive subgroup) of a non-planar regular complex in \mathbb{E}^3 is a 3-dimensional lattice. The following lattices are particularly relevant for us. Let a be a positive real number, let $k = 1, 2$ or 3 , and let $\mathbf{a} := (a^k, 0^{3-k})$, the vector with k components a and $3 - k$ components 0 . Following [21, p.166], we write $\Lambda_{\mathbf{a}}$ for the sublattice of $a\mathbb{Z}^3$ generated by \mathbf{a} and its images under permutation and changes of sign of coordinates. Observe that

$$\Lambda_{\mathbf{a}} = a\Lambda_{(1^k, 0^{3-k})},$$

when $\mathbf{a} = (a^k, 0^{3-k})$. Then $\Lambda_{(1,0,0)} = \mathbb{Z}^3$ is the standard *cubic lattice*; $\Lambda_{(1,1,0)}$ is the *face-centered cubic lattice* (with basis $(1, 1, 0)$, $(-1, 1, 0)$, $(0, -1, 1)$) consisting of all integral vectors with even coordinate sum; and $\Lambda_{(1,1,1)}$ is the *body-centered cubic lattice* (with basis $(2, 0, 0)$, $(0, 2, 0)$, $(1, 1, 1)$).

Occasionally we will appeal to the enumeration of the finite subgroups of $\mathcal{O}(3)$ (see [11]). In particular, under our assumptions on the complex \mathcal{K} , the special group G_* must be one of five possible groups, namely

$$[3, 3], \quad [3, 3]^+, \quad [3, 3]^*, \quad [3, 4] \quad \text{or} \quad [3, 4]^+. \quad (2)$$

Here, $[p, q]$ and $[p, q]^+$, respectively, denote the full symmetry group or rotation subgroup of a Platonic solid $\{p, q\}$, and $[3, 3]^*$ is defined as

$$[3, 3]^* = [3, 3]^+ \cup (-I)[3, 3]^+ \cong [3, 3]^+ \times C_2,$$

where the tetrahedron $\{3,3\}$ is taken to be centered at the origin and $-I$ denotes the central inversion (generating C_2). Note that $[3,3]^*$ is the symmetry group of a common crystal of pyrite (known as fool's gold, due to its resemblance to gold).

We frequently use the following basic fact, which we record without proof.

Lemma 3.8. *Let C be a cube, and let R be the reflection in a plane determined by a pair of opposite face diagonals of C . If S is any rotational symmetry of C of period 4 or 3, respectively, whose rotation axis is not contained in the mirror of R , then R and S generate the full octahedral group $G(C) = [3,4]$ or its full tetrahedral subgroup $[3,3]$.*

Concluding this section, we also observe that the convex hull of the vertices adjacent to a given vertex v of \mathcal{K} is a vertex-transitive convex polygon or polyhedron, P_v (say), in \mathbb{E}^3 and hence has all its vertices on a sphere centered at v . Here, P_v is a polygon if and only if the vertex-figure of \mathcal{K} at v is planar; this can occur even if \mathcal{K} is not a polyhedron (the complex $\mathcal{K}_5(1,2)$ of Section 6.2 is an example). If $v = F_0$, then G_2 stabilizes the vertex of P_v determined by the vertex of F_1 distinct from F_0 .

4 Complexes with face mirrors

In this section we characterize the regular complexes with a non-trivial flag stabilizer G_Φ as the 2-skeletons of regular 4-apeirotopes in \mathbb{E}^3 . We refer to [21, Ch.7F] for the complete enumeration of these apeirotopes. Recall that there are eight regular 4-apeirotopes in \mathbb{E}^3 , two with finite 2-faces and six with infinite 2-faces (see (3) below); they are precisely the discrete faithful realizations of abstract regular polytopes of rank 4 in \mathbb{E}^3 .

Let \mathcal{K} be a regular complex with group $G = G(\mathcal{K})$, and let G_Φ be non-trivial, where again Φ denotes the base flag of \mathcal{K} . Then, by Theorem 3.6, G is (affinely) irreducible, since polyhedra do not have a non-trivial flag stabilizer. More importantly, by Lemma 3.2, \mathcal{K} has face mirrors. In particular, the faces of \mathcal{K} are planar and the only non-trivial element of G_Φ is the reflection R_3 (say) in the plane containing F_2 . Moreover, by Lemma 3.3, G_2 is a dihedral group of order $2r$ containing R_3 .

Recall from Lemma 3.3(a) that G_0 and G_1 each contain two involutions distinct from R_3 whose product is R_3 . Hence, since R_3 is an improper isometry, one must be a proper isometry (a half-turn) and the other an improper isometry (a point reflection or plane reflection). Let R_0 and R_1 , respectively, denote the improper isometries in G_0 and G_1 distinct from R_3 .

First we discuss complexes \mathcal{K} with finite faces. Since G_0 fixes the centers of F_1 and F_2 , the isometry R_0 cannot be a point reflection and hence must be the plane reflection in the perpendicular bisector of F_1 . Similarly, since G_1 fixes F_0 and the center of F_2 , the isometry R_1 is also a plane reflection.

Let R_2 be a plane reflection such that R_2, R_3 are distinguished generators for the

dihedral group $G_2 \cong D_r$ with mirrors inclined at π/r . Then

$$G = \langle G_0, G_1, G_2 \rangle = \langle R_0, R_1, R_2, R_3 \rangle,$$

and G is a discrete irreducible reflection group in \mathbb{E}^3 . We know that the generator R_3 commutes with R_0 and R_1 by Lemma 3.3 (a). Moreover, the mirrors of R_0 and R_2 are perpendicular, and hence R_0 and R_2 commute as well. It follows that the Coxeter diagram associated with the generators R_0, R_1, R_2, R_3 of G is a string diagram. In particular, G must be an infinite group (geometrically) isomorphic to the Coxeter group $[4, 3, 4]$, the symmetry group of the regular cubical tessellation $\{4, 3, 4\}$ in \mathbb{E}^3 (see [5]). Note here that the subgroups $\langle R_0, R_1, R_2 \rangle$ and $\langle R_1, R_2, R_3 \rangle$ each are finite crystallographic (plane) reflection groups in \mathbb{E}^3 .

We now employ a variant of Wythoff's construction to show that \mathcal{K} coincides with the 2-skeleton of the cubical tessellation $\{4, 3, 4\}$ of \mathbb{E}^3 associated with the generators R_0, R_1, R_2, R_3 .

First note that the base vertex F_0 is fixed by G_1 and G_2 , and hence by R_1, R_2 and R_3 . This implies that F_0 is just the base (initial) vertex of $\{4, 3, 4\}$ and that the subgroup generated by R_1, R_2, R_3 is its stabilizer in G . In \mathcal{K} , the orbit of F_0 under $G_0 = \langle R_0, R_3 \rangle$ consists only of F_0 and $F_0 R_0$, so the base edge F_1 of \mathcal{K} is the line segment joining F_0 and $F_0 R_0$. Hence F_1 is just the base edge of $\{4, 3, 4\}$. Similarly, the orbit of F_1 under $\langle G_0, G_1 \rangle = \langle R_0, R_1, R_3 \rangle$ in \mathcal{K} coincides with the orbit of F_1 under the subgroup $\langle R_0, R_1 \rangle$, and hence the base face F_2 of \mathcal{K} is just the base face of $\{4, 3, 4\}$. Finally, then, the vertices, edges and faces of \mathcal{K} are just the images of F_0, F_1 and F_2 under G , and hence \mathcal{K} is the 2-skeleton of $\{4, 3, 4\}$.

Before proceeding with complexes \mathcal{K} with infinite faces, we briefly review the *free abelian apeirotope* or “apeir” construction described in [17, 18]. This construction actually applies to any finite (rational) regular polytope \mathcal{Q} in some euclidean space, but here we only require it for regular polyhedra \mathcal{Q} in \mathbb{E}^3 , where it produces a regular apeirotope of rank 4 in \mathbb{E}^3 with vertex-figure \mathcal{Q} .

Let \mathcal{Q} be a finite regular polyhedron in \mathbb{E}^3 with symmetry group $G(\mathcal{Q}) = \langle \hat{R}_1, \hat{R}_2, \hat{R}_3 \rangle$ (say), where the labeling of the distinguished generators begins at 1 deliberately. Let o be the centroid of the vertex-set of \mathcal{Q} , let w be the initial vertex of \mathcal{Q} , and let \hat{R}_0 denote the reflection in the point $\frac{1}{2}w$. Then there is a regular 4-apeirotope in \mathbb{E}^3 , denoted *apeir* \mathcal{Q} , with $\hat{R}_0, \hat{R}_1, \hat{R}_2, \hat{R}_3$ as the generating reflections of its symmetry group, o as initial vertex, and \mathcal{Q} as vertex-figure. In particular, *apeir* \mathcal{Q} is discrete if \mathcal{Q} is rational (the vertices of \mathcal{Q} have rational coordinates with respect to some coordinate system). The latter limits the choices of \mathcal{Q} to $\{3, 3\}$, $\{3, 4\}$ or $\{4, 3\}$, or their Petrie duals $\{4, 3\}_3$, $\{6, 4\}_3$ or $\{6, 3\}_4$, respectively; the six corresponding 4-apeirotopes are listed in (3).

Now consider complexes \mathcal{K} with infinite (planar) faces, that is, zigzag faces. Now the generator R_0 must be a point reflection in the midpoint of F_1 . In fact, if R_0 was a plane reflection (necessarily in the perpendicular bisector of F_1), the invariance of F_1 and F_2 under R_0 would force F_2 to be a linear apeirogon, which is not possible. On the other

hand, R_1, R_2 and R_3 still are plane reflections, with R_2 and R_3 as before. In fact, if R_1 was a point reflection (necessarily in the point F_0), then again F_2 would have to be a linear apeirogon. Moreover, since R_1 leaves F_0 and F_2 invariant, its plane mirror must necessarily be perpendicular to the plane containing F_2 . Hence, since the latter is the mirror of R_3 , the generators R_1 and R_3 must commute (this also follows from Lemma 3.3(a)). Similarly, the point reflection R_0 commutes with R_2 and R_3 , because its fixed point (the midpoint of F_1) lies on the mirrors of R_2 and R_3 . In addition, the generators R_1, R_2, R_3 all fix F_0 , so that $\langle R_1, R_2, R_3 \rangle$ is a finite, irreducible crystallographic reflection group in \mathbb{E}^3 . This group has a string Coxeter diagram; note here that R_1 and R_2 cannot commute, again since faces are not linear apeirogons. Hence, this group must necessarily be $[3, 3]$, $[3, 4]$ or $[4, 3]$, with R_1, R_2, R_3 the distinguished generators.

Now we can prove that \mathcal{K} is the 2-skeleton of the regular 4-apeirotope $\text{apeir}\{3, 3\}$, $\text{apeir}\{3, 4\}$ or $\text{apeir}\{4, 3\}$ in \mathbb{E}^3 . For convenience we take F_0 to be the origin o , so that the subgroup $\langle R_1, R_2, R_3 \rangle$ is just the standard Coxeter group $[3, 3]$, $[3, 4]$ or $[4, 3]$ with distinguished generators R_1, R_2, R_3 . Set $F'_0 := F_0 R_0$, so that F_1 has vertices F_0 and F'_0 . Then R_0 is the reflection in the point $\frac{1}{2}F'_0$, and the orbit of F'_0 under $\langle R_1, R_2, R_3 \rangle$ is just the vertex set of a copy of $\mathcal{Q} = \{3, 3\}$, $\{3, 4\}$ or $\{4, 3\}$, respectively. Thus the configuration of the mirrors of R_0, R_1, R_2, R_3 is exactly that of the mirrors of the distinguished generators for the symmetry group of the 4-apeirotope $\text{apeir } \mathcal{Q}$. Hence G must be the symmetry group of $\text{apeir } \mathcal{Q}$. Moreover, the initial vertex, $F_0 = o$, is the same in both cases, so the vertex sets of \mathcal{K} and $\text{apeir } \mathcal{Q}$ must be the same as well. The proof that the edges and faces of \mathcal{K} are just those of $\text{apeir } \mathcal{Q}$ follows from the same general argument as for complexes with finite faces.

In summary we have established the following theorem.

Theorem 4.1. *Every regular polygonal complex with a non-trivial flag stabilizer is the 2-skeleton of a regular 4-apeirotope in \mathbb{E}^3 .*

Let \mathcal{P} be a regular 4-apeirotope in \mathbb{E}^3 , and let π denote the Petrie operation on (the vertex-figure of) \mathcal{P} . Recall that, if $\Gamma(\mathcal{P}) = \langle T_0, T_1, T_2, T_3 \rangle$, then π is determined by changing the generators on $\Gamma(\mathcal{P})$ according to

$$\pi : (T_0, T_1, T_2, T_3) \mapsto (T_0, T_1 T_3, T_2, T_3).$$

The 4-apeirotope associated with the new generators on the right is called the Petrie-dual of \mathcal{P} and is denoted by \mathcal{P}^π (see [21, Ch.7F]).

Our above analysis shows that \mathcal{K} is the 2-skeleton of either the cubical tessellation $\{4, 3, 4\}$ if the faces are finite, or the regular 4-apeirotope $\text{apeir}\{3, 3\}$, $\text{apeir}\{3, 4\}$ or $\text{apeir}\{4, 3\}$ if the faces are infinite. This covers four of the eight regular 4-apeirotopes in \mathbb{E}^3 . However, the eight apeirotopes occur in pairs of Petrie-duals and, as we show in the next lemma, the Petrie operation does not affect the 2-skeleton. Thus, all eight apeirotopes actually occur but contribute only four regular complexes.

Lemma 4.2. *If \mathcal{P} is a regular 4-apeirotope in \mathbb{E}^3 , then \mathcal{P} and \mathcal{P}^π have the same 2-skeleton.*

Proof. First recall that, for regular 3-polytopes, the Petrie-dual has the same vertices and edges as the original polytope. Since the vertex-figure of \mathcal{P}^π is just the Petrie-dual of the vertex-figure of \mathcal{P} , the base vertices of \mathcal{P} and \mathcal{P}^π are the same (they are the fixed points of the vertex-figure groups), and hence the vertex sets of \mathcal{P} and \mathcal{P}^π are the same. In a 4-apeirotope, the edges and 2-faces containing a given vertex correspond to the vertices and edges of the vertex-figure at that vertex. Therefore, since the vertex-figures of \mathcal{P} and \mathcal{P}^π at their base vertices have the same vertices and edges, the sets of edges and 2-faces of \mathcal{P} and \mathcal{P}^π containing the base vertex must necessarily also be the same. It follows that \mathcal{P} and \mathcal{P}^π have the same 2-skeleton. \square

Concluding this section we list the eight regular 4-apeirotopes in a more descriptive way in pairs of Petrie-duals using the notation in [21]. The apeirotopes in the top row have square faces, and their facets are cubes or Petrie-Coxeter polyhedra $\{4, 6 \mid 4\}$. All others have zigzag faces, and their facets are blends of the Petrie-duals $\{\infty, 3\}_6$ or $\{\infty, 4\}_4$ of the plane tessellations $\{6, 3\}$ or $\{4, 4\}$, respectively, with the line segment $\{\}$ or linear apeirogon $\{\infty\}$ (see [21, Ch.7E]).

$$\begin{array}{ll}
\{4, 3, 4\} & \{\{4, 6 \mid 4\}, \{6, 4\}_3\} \\
\text{apeir}\{3, 3\} = \{\{\infty, 3\}_6\#\{\}, \{3, 3\}\} & \{\{\infty, 4\}_4\#\{\infty\}, \{4, 3\}_3\} = \text{apeir}\{4, 3\}_3 \\
\text{apeir}\{3, 4\} = \{\{\infty, 3\}_6\#\{\}, \{3, 4\}\} & \{\{\infty, 6\}_3\#\{\infty\}, \{6, 4\}_3\} = \text{apeir}\{6, 4\}_3 \\
\text{apeir}\{4, 3\} = \{\{\infty, 4\}_4\#\{\}, \{4, 3\}\} & \{\{\infty, 6\}_3\#\{\infty\}, \{6, 3\}_4\} = \text{apeir}\{6, 3\}_4
\end{array} \tag{3}$$

Note here that $(\text{apeir } \mathcal{Q})^\pi = \text{apeir}(\mathcal{Q}^\pi)$ for each vertex-figure \mathcal{Q} that occurs. The parameter r counting the number of faces around an edge of the 2-skeleton \mathcal{K} is just the last entry in the Schläfli symbol (the basic symbol $\{p, q, r\}$) of the corresponding 4-apeirotope (or, equivalently, its Petrie dual). Hence, $r = 4, 3, 4$ or 3 , respectively.

In summary we have the following theorem.

Theorem 4.3. *Up to similarity, there are precisely four regular polygonal complexes with a non-trivial flag stabilizer, each given by the common 2-skeleton of two regular 4-apeirotopes in \mathbb{E}^3 which are Petrie duals of each other.*

Now that the regular complexes with non-trivial flag stabilizers have been described, we can restrict ourselves to the enumeration of simply flag-transitive regular complexes.

5 Operations

In this section we discuss operations on the generators of the symmetry group of a regular polygonal complex that allow us to construct new complexes from old. In particular, this will help reduce the number of cases to be considered in the classification.

Let \mathcal{K} be a regular complex in \mathbb{E}^3 , and let \mathcal{K} have a simply flag-transitive (full symmetry) group $G = G(\mathcal{K}) = \langle G_0, G_1, G_2 \rangle$. Then recall from Lemma 3.4(a) that $G_0 = \langle R_0 \rangle$ for some point, line or plane reflection R_0 , that $G_1 = \langle R_1 \rangle$ for some line or plane reflection R_1 , and that G_2 is cyclic or dihedral of order r .

We will define two operations on the group G which replace R_0 or R_1 , respectively, but retain G_2 . They employ elements R of G_2 with the property that R_0R or R_1R , respectively, is an involution. Here we are not assuming that R itself is an involution; however, this will typically be the case.

5.1 Operation λ_0

To begin with, suppose R is an element of G_2 such that R_0R is an involution (if R_0 is a point or plane reflection, this actually forces R to be an involution as well). At the group level we define our first operation λ_0 on the generating subgroups $G_0 = G_0(\mathcal{K})$, $G_1 = G_1(\mathcal{K})$, $G_2 = G_2(\mathcal{K})$ of G by way of

$$\lambda_0 = \lambda_0(R): (R_0, R_1, G_2) \mapsto (R_0R, R_1, G_2). \quad (4)$$

Lemma 5.1. *Let \mathcal{K} be a simply flag-transitive regular complex with group $G = \langle R_0, R_1, G_2 \rangle$, and let R be an element in G_2 such that R_0R is an involution. Then there exists a regular complex, denoted \mathcal{K}^{λ_0} , with the same vertex set and edge set as \mathcal{K} and with the same symmetry group G , such that*

$$\langle R_0R \rangle \subseteq G_0(\mathcal{K}^{\lambda_0}), \quad G_1(\mathcal{K}) = \langle R_1 \rangle \subseteq G_1(\mathcal{K}^{\lambda_0}), \quad G_2(\mathcal{K}) = G_2(\mathcal{K}^{\lambda_0}). \quad (5)$$

The inclusions in (5) are equalities if and only if \mathcal{K}^{λ_0} is simply flag-transitive.

Proof. Let $R'_0 := R_0R$, and let $G'_0 := \langle R_0R \rangle$, $G'_1 := G_1$ and $G'_2 := G_2$. We shall obtain the complex \mathcal{K}^{λ_0} by Wythoff's construction (see [21, Ch.5B]). First note that the subgroup $\langle G'_1, G'_2 \rangle$ has exactly one fixed point, namely the base vertex F_0 of \mathcal{K} . Thus we also take F_0 as base (initial) vertex for \mathcal{K}^{λ_0} . Since both R_0 and R leave the base edge F_1 of \mathcal{K} invariant, the orbit of F_0 under G'_0 consists of the two vertices of F_1 , and hence F_1 can also serve as base edge of \mathcal{K}^{λ_0} . Moreover, the vertex set and edge set, respectively, of the base face F'_2 of \mathcal{K}^{λ_0} are determined by the orbits of F_0 and F_1 under the subgroup $\langle R'_0, R_1 \rangle$. Finally, then, the complex \mathcal{K}^{λ_0} consists of all the vertices, edges and faces obtained as images of F_0 , F_1 and F'_2 under G .

Note that \mathcal{K}^{λ_0} really is a complex. By construction, the graphs of \mathcal{K} and \mathcal{K}^{λ_0} coincide, since their vertex sets and edge sets are the same. In particular, the graph of \mathcal{K}^{λ_0} is connected. Moreover, (possibly) up to a change of edge multiplicities (from single to double, or vice versa), the vertex-figures of \mathcal{K} and \mathcal{K}^{λ_0} at the common base vertex F_0 coincide, since each can be obtained by Wythoff's construction applied to $\langle R_1, G_2 \rangle$ with base vertex $F_0R_0 = F_0R'_0$. In particular, the vertex-figures remain connected. The discreteness follows from our comments made after Definition 2.1. Finally, by the edge-transitivity of

G on \mathcal{K}^{λ_0} and the fact that G_2 remained unchanged, each edge must again be contained in a fixed number of faces; this number is r if \mathcal{K}^{λ_0} is simply flag-transitive. \square

Notice that the operation λ_0 actually also applies (with slight modifications) to regular complexes whose (full symmetry) group is not simply flag-transitive, that is, to the 2-skeletons of regular 4-apertopes. In this more general context, the operation λ_0 becomes invertible and its inverse is associated with the element R^{-1} of G_2 . There are instances in Lemma 5.1 when the group of the resulting complex \mathcal{K}^{λ_0} is no longer simply flag-transitive, and then the more general operation is needed to recover the original complex from it.

For example, if \mathcal{K} is the regular complex whose faces are all the Petrie polygons of all the cubes of the cubical tessellation $\{4, 3, 4\}$ of \mathbb{E}^3 (and the vertices and edges of \mathcal{K} are just those of $\{4, 3, 4\}$), then for a suitable choice of R in G_2 , the new complex \mathcal{K}^{λ_0} is the 2-skeleton of $\{4, 3, 4\}$. In the notation of Section 6.2, $\mathcal{K} = \mathcal{K}_6(1, 2)$. In this example, the vertex-figures of \mathcal{K}^{λ_0} are simple graphs, while those of \mathcal{K} have double edges.

5.2 Operation λ_1

Next we consider the operation λ_1 on the subgroups G_0 , G_1 and G_2 of G defined by

$$\lambda_1 = \lambda_1(R) : (R_0, R_1, G_2) \mapsto (R_0, R_1 R, G_2), \quad (6)$$

where now $R \in G_2$ is such that $R_1 R$ is an involution. Again we shall concentrate on complexes with a simply flag-transitive group, although the operation applies (with slight modifications) more generally to arbitrary regular complexes. Now we have

Lemma 5.2. *Let \mathcal{K} be a simply flag-transitive regular complex with group $G = \langle R_0, R_1, G_2 \rangle$, and let R be an element in G_2 such that $R_1 R$ is an involution. Then there exists a regular complex, denoted \mathcal{K}^{λ_1} and again simply flag-transitive, with the same vertex set and edge set as \mathcal{K} and with the same group G , such that*

$$G_0(\mathcal{K}) = \langle R_0 \rangle = G_0(\mathcal{K}^{\lambda_1}), \quad \langle R_1 R \rangle = G_1(\mathcal{K}^{\lambda_1}), \quad G_2(\mathcal{K}) = G_2(\mathcal{K}^{\lambda_0}). \quad (7)$$

Proof. The proof follows the same pattern as the proof of Lemma 5.1. As before, we employ Wythoff's construction with initial vertex F_0 to construct \mathcal{K}^{λ_1} . Again, the edge graphs of \mathcal{K} and \mathcal{K}^{λ_1} are the same; note here that the element $R'_1 := R_1 R = R^{-1} R_1$ maps F_1 to $F_1 R_1$, which is the other edge of F_2 containing F_0 .

The connectedness and discreteness of \mathcal{K}^{λ_1} are derived as before. The vertex-figure of \mathcal{K}^{λ_1} at F_0 is obtained from Wythoff's construction with group $\langle R_1 R, G_2 \rangle = \langle R_1, G_2 \rangle$ and initial vertex $F_0 R_0$, and hence is the same as the vertex-figure of \mathcal{K} at F_0 , again (possibly) up to a change of edge multiplicities. In particular, the vertex-figures remain connected.

However, unlike in the previous case, the complex \mathcal{K}^{λ_1} cannot have a non-trivial flag stabilizer. In fact, when applied to a regular complex with a non-trivial flag stabilizer $\langle R_3 \rangle$, the operation λ_1 must necessarily be the Petrie operation (on the vertex-figure) of

the corresponding regular 4-apeirotope (there is just one possible choice for R , namely R_3) and hence, by Lemma 4.2, must leave the 2-skeleton invariant. Now suppose \mathcal{K}^{λ_1} has a non-trivial flag stabilizer and hence is the 2-skeleton of a regular 4-apeirotope. Then, since λ_1 is invertible, \mathcal{K} can be recovered from \mathcal{K}^{λ_1} by the inverse of λ_1 , which necessarily must be the Petrie operation on (the apeirotope determined by) \mathcal{K}^{λ_1} . It follows that the original complex \mathcal{K} must have had a non-trivial flag stabilizer as well. Thus \mathcal{K}^{λ_1} is simply flag-transitive.

Finally, by Lemma 3.4(b), the number of faces around an edge of \mathcal{K}^{λ_1} is r , since $G_2(\mathcal{K}) = G_2(\mathcal{K}^{\lambda_1})$. \square

5.3 Mirror vectors $(2, k)$ from $(0, k)$ by way of λ_0

We now discuss some particularly interesting applications of λ_0 and λ_1 . Once again we restrict ourselves to regular complexes \mathcal{K} with simply flag-transitive symmetry groups. We call the vector $(\dim(R_0), \dim(R_1))$ the *mirror vector* of \mathcal{K} , where $\dim(R_i)$ is the dimension of the mirror (fixed point set) of the symmetry R_i for $i = 0, 1$. Here, to keep notation simple, we suppress information about the subgroup G_2 , although this is not to imply that the structure of G_2 is irrelevant. We sometimes indicate the mirror vector explicitly and denote a complex \mathcal{K} with mirror vector (j, k) by $\mathcal{K}(j, k)$.

If \mathcal{K} is a regular polyhedron, then G_2 is generated by a (point, line or plane) reflection R_2 and we refer to the vector $(\dim(R_0), \dim(R_1), \dim(R_2))$ as the *complete mirror vector* of \mathcal{K} . (In [21, Ch.7E], this vector was called the dimension vector.) In the context of the present paper we usually take the enumeration of regular polyhedra for granted and concentrate on complexes that are not polyhedra.

We begin with the operation λ_0 . Let \mathcal{K} be a regular complex with mirror vector $(2, k)$ for some $k = 1, 2$; then R_0 is the reflection in the perpendicular bisector of F_1 . (Recall here that the case $k = 0$ cannot occur by Lemma 3.4(a).) If $R \in G_2$ is a half-turn, then its mirror (axis) must necessarily be the line through F_1 , so R_0R must necessarily be the point reflection in the midpoint of F_1 . It follows that the corresponding complex \mathcal{K}^{λ_0} has mirror vector $(0, k)$. Conversely, let \mathcal{K} be a regular complex with mirror vector $(0, k)$ for some $k = 1, 2$. If $R \in G_2$ is a half-turn, then its mirror must necessarily contain the mirror of R_0 (the midpoint of F_1), so R_0R must necessarily be the plane reflection in the perpendicular bisector of F_1 . Hence the corresponding complex \mathcal{K}^{λ_0} has mirror vector $(2, k)$. In either case, λ_0 is an involutory operation.

The next lemma allows us to derive the enumeration of regular polygonal complexes with $\dim(R_0) = 2$ from the enumeration of those with $\dim(R_0) = 0$.

Lemma 5.3. *Let \mathcal{K} be an infinite simply flag-transitive regular complex with an affinely irreducible group and mirror vector $(2, k)$ for some $k = 1, 2$. Then $G_2(\mathcal{K})$ contains a half-turn R . In particular, the corresponding complex \mathcal{K}^{λ_0} , with $\lambda_0 = \lambda_0(R)$, is a regular complex with mirror vector $(0, k)$, and $\mathcal{K} = (\mathcal{K}^{\lambda_0})^{\lambda_0}$.*

Proof. Appealing to the enumeration of finite subgroups of isometries of \mathbb{E}^3 , we first observe that the special subgroup G_* of G must be a subgroup of the octahedral group $[3, 4]$ (see (2)); note here that G_* is linearly irreducible and that G is an infinite group. Now suppose that G_2 does not contain a half-turn. Then G_2 must be a cyclic group C_3 or dihedral group D_3 and hence must contain a 3-fold rotation with axis perpendicular to the plane mirror of R_0 . On the other hand, the octahedral group does not contain a 3-fold rotation with an axis perpendicular to a plane reflection mirror. Thus G_2 must contain a half-turn. Now the lemma follows from our previous considerations. \square

Recall from Theorem 3.6 that a regular complex in \mathbb{E}^3 with a reducible group must be a planar or blended polyhedron. Of these polyhedra, only the three planar tessellations $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$ have mirror vectors of type $(2, k)$ (in fact, of type $(2, 2)$), and then G_2 is generated by a plane reflection and does not contain a half-turn. In other words, our irreducibility assumption in Lemma 5.3 really only eliminates these three choices for \mathcal{K} .

Furthermore, notice for Lemma 5.3 that, vice versa, we may not generally be able to similarly obtain a given regular complex with mirror vector $(0, k)$ from one with vector $(2, k)$. In fact, the octahedral group does contain 3-fold rotations about axes passing through the mirror of a point reflection (namely, the central involution), so G_2 might not contain a half-turn.

5.4 Mirror vector $(0, k)$ from $(1, k)$ by way of λ_0

Now let \mathcal{K} be a regular complex with mirror vector $(0, k)$ for some $k = 1, 2$ (as before, the case $k = 0$ cannot occur); then R_0 is the point reflection in the midpoint of F_1 . If $R \in G_2$ is a plane reflection, then its mirror must necessarily contain F_1 (and hence the mirror of R_0), so R_0R must be the half-turn about the line perpendicular to the mirror of R . Then the corresponding complex \mathcal{K}^{λ_0} has mirror vector $(1, k)$. Conversely, let \mathcal{K} be a regular complex with mirror vector $(1, k)$ for some $k = 1, 2$; then R_0 is a half-turn about an axis perpendicular to the line through F_1 and passing through the midpoint of F_1 . If $R \in G_2$ is a plane reflection whose mirror is perpendicular to the mirror of R_0 , then R_0R is a point reflection in the midpoint of F_1 . Now the corresponding complex \mathcal{K}^{λ_0} has mirror vector $(0, k)$. In either case, λ_0 is involutory.

The following lemma allows us to deduce the enumeration of regular complexes with $\dim(R_0) = 0$ and a dihedral subgroup G_2 from the enumeration of those with $\dim(R_0) = 1$ (and a dihedral group G_2).

Lemma 5.4. *Let \mathcal{K} be a simply flag-transitive regular complex with an affinely irreducible group, a dihedral subgroup G_2 , and mirror vector $(0, k)$ for some $k = 1, 2$. Then, for any plane reflection $R \in G_2$, the corresponding complex \mathcal{K}^{λ_0} , with $\lambda_0 = \lambda_0(R)$, is a regular complex with mirror vector $(1, k)$. In particular, $\mathcal{K} = (\mathcal{K}^{\lambda_0})^{\lambda_0}$.*

Proof. All we need to say here is that R_0R is the half-turn about the line through the midpoint of F_1 perpendicular to the mirror of R . Then Lemma 5.1 applies. \square

As in the previous case, we may not generally be able to obtain a given regular complex with mirror vector $(1, k)$ from a complex with vector $(0, k)$, unless the axis of the half-turn R_0 is perpendicular to the mirror of a plane reflection in G_2 . However, the latter condition is not guaranteed.

It is not hard to see that besides the scenarios described in Lemmas 5.3 and 5.4 there are only three other possible choices of R in which λ_0 can be applied. First, if \mathcal{K} has mirror vector $(2, k)$ and G_2 is dihedral, then trivially any plane reflection R in G_2 has mirror perpendicular to the mirror of R_0 and leads to a new complex \mathcal{K}^{λ_0} with mirror vector $(1, k)$. Second, and conversely, if \mathcal{K} has mirror vector $(1, k)$, G_2 is dihedral, and G_2 contains a plane reflection R whose mirror contains the axis of R_0 , then we arrive at a new complex \mathcal{K}^{λ_0} with mirror vector $(2, k)$. For these two choices, $\mathcal{K} = (\mathcal{K}^{\lambda_0})^{\lambda_0}$. Third, if \mathcal{K} has mirror vector $(1, k)$ and R is a rotation in G_2 , then R_0R is the half-turn about an axis contained in the perpendicular bisector of F_1 , and hence \mathcal{K}^{λ_0} is again a complex with mirror vector $(1, k)$. For the third choice, note that R need not be an involution and thus λ_0 need not be involutory.

5.5 Mirror vector $(k, 1)$ from $(k, 2)$ by way of λ_1

Next we turn to applications of λ_1 . Let \mathcal{K} be a regular complex with affinely irreducible symmetry group G , and let \mathcal{K} be simply flag-transitive. Assume that the mirror vector of \mathcal{K} is of the form $(k, 1)$ for some $k = 0, 1, 2$, and that G_2 contains a plane reflection R whose mirror contains the axis of the half-turn R_1 . Then R_1R is the reflection in the plane that is perpendicular to the mirror of R and contains the axis of R_1 . It follows that the corresponding regular complex \mathcal{K}^{λ_1} has mirror vector $(k, 2)$. Conversely, if \mathcal{K} has mirror vector $(k, 2)$ and G_2 contains a plane reflection R whose mirror is perpendicular to the plane mirror of R_1 , then R_1R is the half-turn whose axis is the intersection of the two mirrors. Now the complex \mathcal{K}^{λ_1} has mirror vector $(k, 1)$. Once again, in either case the operation λ_1 is involutory.

In summary, appealing to Lemma 5.2, we now have the following result that allows us to derive the enumeration of certain regular complexes with $\dim(R_1) = 1$ from the enumeration of those with $\dim(R_1) = 2$.

Lemma 5.5. *Let \mathcal{K} be a simply flag-transitive regular complex with an affinely irreducible group and mirror vector $(k, 1)$ for some $k = 0, 1, 2$. Assume also that G_2 contains a plane reflection R whose mirror contains the axis of the half-turn R_1 . Then the corresponding complex \mathcal{K}^{λ_1} , with $\lambda_1 = \lambda_1(R)$, is a regular complex with mirror vector $(k, 2)$. In particular, $\mathcal{K} = (\mathcal{K}^{\lambda_1})^{\lambda_1}$.*

It can be shown that the above analysis exhausts all possible choices for elements R in G_2 that can lead to a new complex \mathcal{K}^{λ_1} via the corresponding operations $\lambda_1 = \lambda_1(R)$.

6 Complexes with finite faces and mirror vector $(1, 2)$

In this section and the next, we enumerate the simply flag-transitive regular complexes with mirror vector $(1, 2)$. Here we begin with complexes with finite faces. Throughout, let \mathcal{K} be an infinite simply flag-transitive regular complex with an affinely irreducible group $G = \langle R_0, R_1, G_2 \rangle$.

6.1 The special group

As we remarked earlier, if a regular complex \mathcal{K} has finite faces, then these faces are necessarily planar or skew regular polygons. If additionally \mathcal{K} has mirror vector $(1, 2)$, then R_0 is a line reflection (half-turn) and R_1 a plane reflection whose mirrors intersect at the center of the base face F_2 . Let E_1 denote the perpendicular bisector of the base edge F_1 . Then the subgroup G_2 leaves the line through F_1 pointwise fixed and acts faithfully on E_1 . The product R_0R_1 is a rotatory reflection that leaves the plane E_2 through the midpoints of the edges of F_2 invariant, and its period is just the number of vertices of F_2 . Note that E_2 is perpendicular to the plane mirror of R_1 and necessarily contains the axis of R_0 . Since the axis of R_0 also lies in E_1 and $E_1 \neq E_2$, it must necessarily coincide with $E_1 \cap E_2$.

Throughout, it is convenient to distinguish the following two alternative scenarios for R_0 and G_2 and treat them as separate cases.

- (A) The axis of R_0 is contained in the mirror of a plane reflection, R_2 (say), in G_2 .
- (B) The axis of R_0 is not contained in the mirror of a plane reflection in G_2 .

If G_2 is dihedral, then Lemma 3.4(b) tells us that r is even and $G_2 \cong D_{r/2}$. Case (A) can only occur if G_2 is dihedral, and then R_0R_2 is the plane reflection with mirror E_1 and $\langle R_0, G_2 \rangle$ is isomorphic to $D_{r/2} \times C_2$, with the factor C_2 generated by R_0R_2 . In dealing with case (A), R_2 will always denote the plane reflection in G_2 whose mirror contains the axis of R_0 . On the other hand, in case (B), the subgroup G_2 may be cyclic or dihedral. If G_2 is dihedral, then the axis of R_0 must lie “halfway” (in E_1) between the plane mirrors of two basic generators of G_2 inclined at an angle $2\pi/r$; in fact, by Lemma 3.1(e), G_2 must be invariant under conjugation by R_0 . In this case, $\langle R_0, G_2 \rangle$ is isomorphic to D_r and contains a rotatory reflection of order r .

Recall our notation for the elements of the special group G_* of G (see Section 3). In particular, if R is a general element of G , then R' is its image in G_* . Similarly, if E is a plane, then E' will denote its translate through the origin o .

It is convenient to assume that o is the base vertex of \mathcal{K} . Then G_* contains R_1 as an element and G_2 as a subgroup; in fact, $R_1 = R'_1$ and $R = R'$ for each $R \in G_2$ (for short, $G_2 = G'_2$), since R_1 and R fix the base vertex, o . It follows that the vertex-figure group $\langle R_1, G_2 \rangle$ of \mathcal{K} at o is a subgroup of G_* .

In the present context, G_* is a finite irreducible crystallographic subgroup of $\mathcal{O}(3)$ that contains a rotatory reflection, R'_0R_1 , whose invariant plane, E'_2 , is perpendicular to

the mirror of a plane reflection, R_1 , and whose period is just that of R_0R_1 . This trivially excludes the groups $[3, 3]^+$ and $[3, 4]^+$ as possibilities for G_* (see (2)). Moreover, we can also eliminate $[3, 3]^*$ on the following grounds. In fact, if we place the vertices of a tetrahedron $\{3, 3\}$ at alternating vertices of a cube that is centered at o and has its faces parallel to the coordinate planes, then the possible mirrors for plane reflections in $[3, 3]^*$ are just the coordinate planes; however, these are not perpendicular to invariant planes of rotatory reflections in $[3, 3]^*$. (Note here that the rotary reflection $-I$, of period 2, cannot occur as R'_0R_1 .) Thus G_* must be one of the groups $[3, 3]$ or $[3, 4]$.

Next observe that $[3, 3]$ does indeed occur as the special group of an infinite regular polyhedron with finite faces and mirror vector $(1, 2)$; however, there is just one polyhedron of this kind, namely $\{6, 6\}_4$ (see [21, p.225]). On the other hand, the following considerations will exclude $[3, 3]$ as a special group for complexes \mathcal{K} that are not polyhedra. Suppose \mathcal{K} has $[3, 3]$ as its special group but \mathcal{K} is not a polyhedron (that is, $r \geq 3$).

Again it is convenient to place the vertices of $\{3, 3\}$ at alternating vertices of a cube that is centered at o and has its faces parallel to the coordinate planes. Then R'_0 is a half-turn about a coordinate axis, R'_0R_1 is a rotatory reflection of period 4 with invariant plane E'_2 containing this axis, and R_1 is a reflection in a plane perpendicular to E'_2 . Since R'_0 and R_1 cannot commute, the mirror of R_1 cannot be a coordinate plane.

We now consider the subgroup G_2 of G_* , bearing in mind that $r \geq 3$. Then G_2 must contain a non-trivial rotation S about an axis orthogonal to the axis of R'_0 . Hence, since $G_* = [3, 3]$, this must necessarily be a half-turn about a coordinate axis distinct from the axis of R'_0 . Moreover, since G_* is linearly irreducible, we must have $E'_1 \neq E'_2$, that is, E'_1 is perpendicular to E'_2 and the rotation axis of S lies in E'_2 . Appealing again to irreducibility (or the fact that \mathcal{K} is not a polyhedron), we see that G_2 must be a dihedral group of order 4 and hence contain two plane reflections whose mirrors are perpendicular and intersect in the rotation axis of S ; moreover, in case (A), the group G_* also contains the reflection in E'_1 . However, since $[3, 3]$ does not contain three plane reflections with mutually perpendicular mirrors, this rules out case (A) and only leaves case (B) with a configuration of mirrors and axes as in Figure 1, with R'_0 and R_1 corresponding to $(T_0T_3)'$ and T_1 , respectively, and the two reflections in G_2 (with mirrors meeting at the axis of S) corresponding to T_2 and $T_3T_2T_3$. Now, to complete the proof we require the following lemma, which is of interest in its own right.

Lemma 6.1. *Let $[4, 3, 4] = \langle T_0, T_1, T_2, T_3 \rangle$ be the symmetry group of the cubical tessellation $\{4, 3, 4\}$ of \mathbb{E}^3 , where T_0, T_1, T_2, T_3 are the distinguished generators (see Figure 1). Let $H := \langle T_0T_3, T_1, T_2, T_3T_2T_3 \rangle$. Then H is a subgroup of $[4, 3, 4]$ of index 2 and acts simply flag-transitively on the 2-skeleton of $\{4, 3, 4\}$. In particular, the 2-skeleton of $\{4, 3, 4\}$ can be recovered from H by Wythoff's construction.*

Proof. Figure 1 shows the eight cubes of the cubical tessellation $\{4, 3, 4\}$ that meet at the base vertex o (labeled 0). The generators T_0, T_1, T_2, T_3 and their images $T'_0, T'_1 = T_1$,

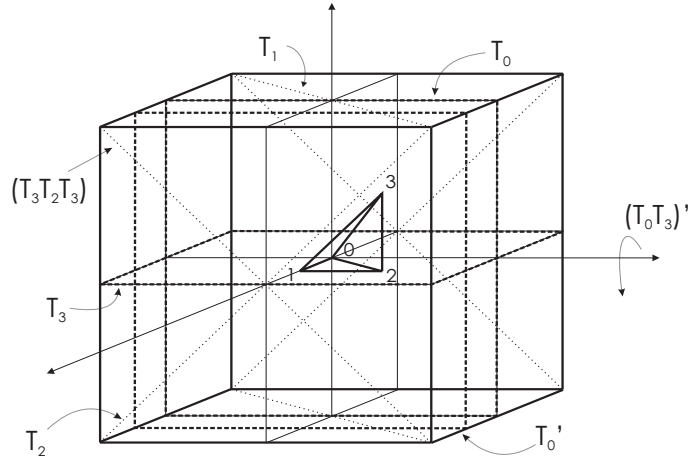


Figure 1: Generators of the special group of H .

$T'_2 = T_2$, $T'_3 = T_3$ in the special group are indicated. Here, T_j is the reflection in the plane of the small fundamental tetrahedron opposite to the vertex labeled j , for $j = 0, 1, 2, 3$.

The two elements $T_1(T_2T_3)^2T_1$ and T_0T_3 are half-turns about parallel axes (in Figure 1, the y -axis and the line through the points labeled 1, 2) and their product is a basic translation (along the x -axis) in H , whose conjugates under H generate the full translation subgroup of $[4, 3, 4]$. On the other hand, H must be a proper subgroup of $[4, 3, 4]$; in fact, its special group is a subgroup $[3, 3]$ of the special group $[3, 4]$ of $[4, 3, 4]$ generated by $(T_0T_3)'$, T_1 , T_2 , $T_3T_2T_3$. Bearing in mind that $[4, 3, 4]$ is the semi-direct product of its translation subgroup by its vertex-figure group $[3, 4]$, it now follows that H must have index 2. The coset of H in $[4, 3, 4]$ distinct from H is T_3H .

In order to recover the 2-skeleton of $\{4, 3, 4\}$ from H define subgroups $H_0 := \langle T_0T_3 \rangle$, $H_1 := \langle T_1 \rangle$ and $H_2 := \langle T_2, T_3T_2T_3 \rangle$. Then the base vertex o of $\{4, 3, 4\}$ is fixed by H_1 and H_2 , and the base edge of $\{4, 3, 4\}$ (through the points labeled 0 and 1) is fixed by H_2 . Applying Wythoff's construction with group H (with distinguished subgroups H_0 , H_1 , H_2 as indicated) and with initial vertex o , then yields a complex with its vertices, edges and 2-faces among those of $\{4, 3, 4\}$. However, since T_3H is the only non-trivial coset of H and the element T_3 leaves the base vertex, base edge and base 2-face of $\{4, 3, 4\}$ invariant, the images of the latter under the full group $[4, 3, 4]$ are just those under the subgroup H . Hence, Wythoff's construction applied to H gives the full 2-skeleton of $\{4, 3, 4\}$. Moreover, since T_3 is the reflection in the plane through the base 2-face of $\{4, 3, 4\}$, it stabilizes the base flag of the 2-skeleton and hence is the only non-trivial element in the flag stabilizer. Since T_3 is not in H , it follows that H must be simply flag-transitive. \square

We now complete the proof that $[3, 3]$ cannot occur as a special group $G_* = \langle R'_0, R_1, G_2 \rangle$ if \mathcal{K} is not a polyhedron. Recall that we were in case (B). As noted earlier, the generators

R'_0, R_1 and the two reflections in G_2 can be viewed as appropriate elements of the special group H_* of the simply flag-transitive subgroup H of the symmetry group of the 2-skeleton of $\{4, 3, 4\}$ described in Lemma 6.1 and depicted in Figure 1. In particular, we may identify the base vertex and base edge of \mathcal{K} with those of the 2-skeleton of $\{4, 3, 4\}$; recall here that the axis of S is simply the line through the base edge of $\{4, 3, 4\}$. A straightforward application of Wythoff's construction then shows that \mathcal{K} must indeed coincide with the 2-skeleton of $\{4, 3, 4\}$, contradicting our basic assumption that \mathcal{K} is simply flag-transitive (recall that the latter means that the *full* symmetry group is simply flag-transitive).

In summary, we have established the following

Lemma 6.2. *Let \mathcal{K} be a simply flag-transitive regular complex with finite faces and mirror vector $(1, 2)$, and let \mathcal{K} not be the polyhedron $\{6, 6\}_4$. Then $G_* = [3, 4]$.*

6.2 The complexes associated with $[3, 4]$

As before, let \mathcal{K} be a simply flag-transitive regular complex \mathcal{K} with finite faces and mirror vector $(1, 2)$. Recall our standing assumption that \mathcal{K} is infinite and G is irreducible. Suppose the special group G_* of \mathcal{K} is the octahedral group $[3, 4]$. From the previous lemma we know that this only eliminates the polyhedron $\{6, 6\}_4$. We further assume that the base vertex F_0 of \mathcal{K} is the origin o , so in particular $R'_1 = R_1$ and $G'_2 = G_2$. The vertex v of the base edge F_1 of \mathcal{K} distinct from F_0 is called the *twin vertex* of \mathcal{K} (with respect to the base flag).

First observe that G_2 must contain a non-trivial rotation. Otherwise, G_2 is generated by a single plane reflection and so has order 2. Furthermore, \mathcal{K} must then be a (pure) regular polyhedron with complete dimension vector $(1, 2, 2)$ (see [21, p.225]); however, no such polyhedron exists. Thus G_2 contains a rotation, S (say), that generates its non-trivial rotation subgroup. Note that, as a vector, the twin vertex spans the rotation axis of S passing through o and v .

As the reference figure for the action of the special group G_* we take the cube $C := \{4, 3\}$ with vertices $(\pm 1, \pm 1, \pm 1)$, so that C is centered at o and has faces parallel to the coordinate planes. Its group $[3, 4]$ contains six rotatory reflections of period 4 and eight rotatory reflections of period 6. Each rotatory reflection of period 4 is given by a rotation by $\pm\pi/2$ about a coordinate axis, followed by a reflection in the coordinate plane perpendicular to the axis. Each rotatory reflection of period 6 is given by a rotation by $\pm\pi/3$ about a main diagonal of C , followed by a reflection in the plane through o perpendicular to the diagonal.

We break the discussion down into two cases, Case I and Case II, according as $R'_0 R_1$ is a rotatory reflection of period 4 or 6, or, equivalently, \mathcal{K} has square faces or hexagonal faces. (It is convenient here to use the term “square” to describe a regular quadrangle, such as the Petrie polygon of $\{3, 3\}$.) Recall that E'_1 denotes the plane through o parallel to the perpendicular bisector E_1 of F_1 (on which G_2 acts faithfully), and that E'_2 is the invariant plane of $R'_0 R_1$ and contains the axis $E'_1 \cap E'_2$ of the half-turn R'_0 .

Case I: Square faces

Suppose $R'_0 R_1$ is a rotatory reflection of period 4 with invariant plane E'_2 given by the xy -plane (say). Then, up to conjugacy in $[3, 4]$, there are exactly two possible choices for the half-turn R'_0 , namely R'_0 either rotates about the center of a face of C (a coordinate axis) or about the midpoint of an edge of C .

For the sake of simplicity, when we claim uniqueness for the choice of certain elements within the special group or of mirrors or fixed point sets of such elements, we will usually omit any qualifying statements such as “up to conjugacy” or “up to congruence”. Throughout, these qualifications are understood.

Case Ia: R'_0 rotates about the center of a face of C

Suppose R'_0 rotates about the y -axis (say). Since the mirror of R_1 is a plane perpendicular to E'_2 and the elements R'_0 and R_1 do not commute, there is only one choice for R_1 , namely the reflection in the plane $y = x$. We next consider the possible choices for S and E'_1 . Clearly, $E'_1 \neq E'_2$, since otherwise E'_2 is an invariant subspace for the (irreducible) group G_* . Bearing in mind that E'_1 must contain the rotation axis of R'_0 , this leaves the following two possibilities for E'_1 .

First, suppose E'_1 is perpendicular to E'_2 . Then E'_1 must be the yz -plane. We can eliminate this possibility as follows.

Suppose G_2 is cyclic of order r . Then $r \neq 2$, once more since otherwise E'_2 is an invariant subspace of G_* . If $r > 2$, then necessarily $r = 4$ and the twin vertex v (invariant under S) is of the form $v = (a, 0, 0)$ for some non-zero parameter a . In particular, this forces the square faces of \mathcal{K} to be planar and parallel to the coordinate planes. In fact, Wythoff's construction shows in this case that \mathcal{K} must be the 2-skeleton of the cubical tessellation $\{4, 3, 4\}$ with vertex-set $a\mathbb{Z}^3$. Moreover, G must be the full group $[4, 3, 4]$, since its full translation subgroup and special group are the same as those of $\{4, 3, 4\}$; note here that the translation by v belongs to G (the axes of the half-turns $R_1 S^2 R_1$ and R_0 , respectively, are the lines through o and $\frac{1}{2}v$ parallel to the y -axis, so $R_1 S^2 R_1 R_0$ is the translation by v). All this contradicts our basic assumption that \mathcal{K} is simply flag-transitive.

Next consider the possibility that G_2 is dihedral of order r . Then $r = 4$ or 8 and we are in case (A) or (B), depending on whether or not the axis of R_0 lies in the mirror of a plane reflection of G_2 (see Section 6.1). In case (A) we cannot have $r = 4$, once again on account of the irreducibility of G_* . In case (A) with $r = 8$, the resulting group G is a supergroup of the group discussed in the previous paragraph. In particular, the twin vertex is of the form $v = (a, 0, 0)$ and \mathcal{K} has face mirrors; for example, F_2 lies in the xy -plane, which is the mirror of the reflection R_2 in G_2 . Thus \mathcal{K} is the 2-skeleton of $\{4, 3, 4\}$ and $G = [4, 3, 4]$, again a contradiction to our assumptions on \mathcal{K} . In any case, case (A) with $r = 8$ can be ruled out and we are left with case (B) with $r = 4$. There is just one possible configuration, namely a pair of generating reflections of G_2 with mirrors given by the planes $z = y$ and $z = -y$. This is exactly the mirror configuration described

in Lemma 6.1 and depicted in Figure 1. However, now R'_0 , R_1 and G_2 all preserve the two sets of alternating vertices of C , so $G_* = [3, 3]$, not $[3, 4]$, allowing us to eliminate case (B) as well.

Second, suppose E'_1 is not perpendicular to E'_2 . Then E'_1 is the plane $z = -x$ (say) and S is a half-turn about the midpoint $(1, 0, 1)$ of an edge of C . In any case, $r = 2$ or 4 and the twin vertex has the form $v = (a, 0, a)$ with $a \neq 0$.

If $r = 2$ (and hence G_2 is cyclic), the complex \mathcal{K} is a polyhedron with skew square faces and planar hexagonal vertex-figures. In particular, \mathcal{K} is the polyhedron $\{4, 6\}_6$, since its complete dimension vector is $(1, 2, 1)$ and its special group is $[3, 4]$ (see [21, p.225]). Here the square faces of \mathcal{K} are inscribed into three quarters of all cubes of a cubical tessellation in \mathbb{E}^3 .

If $r = 4$, then G_2 is dihedral and case (B) cannot occur. On the other hand, in case (A), the group G_2 is generated by the reflections R_2, \widehat{R}_2 in the planes $z = x$ and $y = 0$, respectively, and the mirror of R_2 contains the axis of R'_0 (see Figure 2). Then the set of generators $R_0, R_1, R_2, \widehat{R}_2$ of G is given by

$$\begin{aligned} R_0: \quad (x, y, z) &\mapsto (-x, y, -z) + (a, 0, a), \\ R_1: \quad (x, y, z) &\mapsto (y, x, z), \\ R_2: \quad (x, y, z) &\mapsto (z, y, x), \\ \widehat{R}_2: \quad (x, y, z) &\mapsto (x, -y, z), \end{aligned} \tag{8}$$

with $a \neq 0$. Now we obtain a legitimate regular complex, denoted $\mathcal{K}_1(1, 2)$, with skew square faces, four around each edge. (Recall our convention to label regular complexes with their mirror vectors.)

The vertex-set of $\mathcal{K}_1(1, 2)$ consists of the images of the base vertex o under G and coincides with the lattice $\Lambda_{(a, a, 0)}$. The faces of $\mathcal{K}_1(1, 2)$ are Petrie polygons of tetrahedra inscribed at alternating vertices of cubes in the cubical tessellation with vertex-set $a\mathbb{Z}^3$. In each cube, $\mathcal{K}_1(1, 2)$ takes edges of just one of the two possible tetrahedra; the Petrie polygons of this tetrahedron determine three faces of $\mathcal{K}_1(1, 2)$, forming a finite subcomplex $\{4, 3\}_3$ (the Petrie dual of $\{3, 3\}$), such that every edge lies in exactly two of them. Thus every edge of $\mathcal{K}_1(1, 2)$ is surrounded by four faces, so that each cube in a pair of adjacent cubes contributes exactly two faces to the unique edge of $\mathcal{K}_1(1, 2)$ contained in their intersection. In particular, the vertices of the base face F_2 are the images of o under $\langle R_0, R_1 \rangle$ and are given by $o, v = (a, 0, a), (a, a, 0), (0, a, a)$, occurring in this order. Moreover, the vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_1(1, 2)$ at o is the octahedral group $[3, 4]$, occurring here with (standard) generators R_2, R_1, \widehat{R}_2 . The vertex-figure of $\mathcal{K}_1(1, 2)$ at o is the (simple) edge graph of the cuboctahedron with vertices $(\pm a, \pm a, 0), (0, \pm a, \pm a), (\pm a, 0, \pm a)$; in fact, each edge of this graph corresponds to a face of $\mathcal{K}_1(1, 2)$ with vertex o , and vice versa.

Case Ib: R'_0 rotates about the midpoint of an edge of C

Recall that the rotatory reflection $R'_0 R_1$ of period 4 has the xy -plane as its invariant

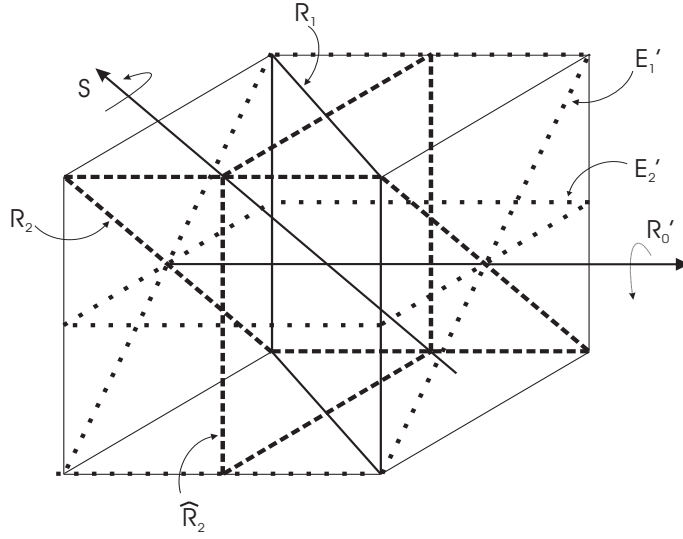


Figure 2: The special group of the complex $\mathcal{K}_1(1, 2)$

plane E_2' . Here we may assume that R_0' is the half-turn about the line through o and the midpoint $(1, 1, 0)$ of an edge of C , and that R_1 is the reflection in a coordinate plane perpendicular to E_2' , the plane $y = 0$ (say). Since E_1' must contain the axis of R_0' , there are two possible choices for E_1' , namely either E_1' is perpendicular to E_2' or to a main diagonal of C ; note here that we must have $E_1' \neq E_2'$, since G is irreducible. We show that the first possibility cannot occur and that the second contributes a new regular complex.

First, suppose E_1' is perpendicular to E_2' . Then E_1' is the plane $y = x$ (say), and G_2 is either cyclic of order 2 or dihedral of order 4. In either case, G_* would be reducible, with invariant subspace E_2' . Thus E_1' cannot be perpendicular to E_2' .

Second, suppose E_2' is perpendicular to a main diagonal of C , the diagonal passing through the pair of antipodal vertices $\pm(1, -1, 1)$ (say). Then $r = 3$ or 6 , and G_2 is either cyclic of order 3 or dihedral of order 6, the latter necessarily occurring here as case (B). In any case, the twin vertex of \mathcal{K} is of the form $v = (a, -a, a)$ with $a \neq 0$.

Suppose G_2 is cyclic of order 3 and is generated by a 3-fold rotation S about the main diagonal of C passing through $\pm(1, -1, 1)$. Then the generators R_0, R_1, S of G are given by

$$\begin{aligned} R_0: \quad (x, y, z) &\mapsto (y, x, -z) + (a, -a, a), \\ R_1: \quad (x, y, z) &\mapsto (x, -y, z), \\ S: \quad (x, y, z) &\mapsto (-y, -z, x), \end{aligned} \tag{9}$$

with $a \neq 0$ (see Figure 3). This determines a new regular complex, denoted $\mathcal{K}_2(1, 2)$, with skew square faces, three around each edge.

The vertex-set of $\mathcal{K}_2(1, 2)$ is the lattice $\Lambda_{(a, a, a)}$. Relative to the cubical tessellation of \mathbb{E}^3 with vertex set $(a, a, a) + 2a\mathbb{Z}^3$, a typical square face of $\mathcal{K}_2(1, 2)$ has its four vertices, in

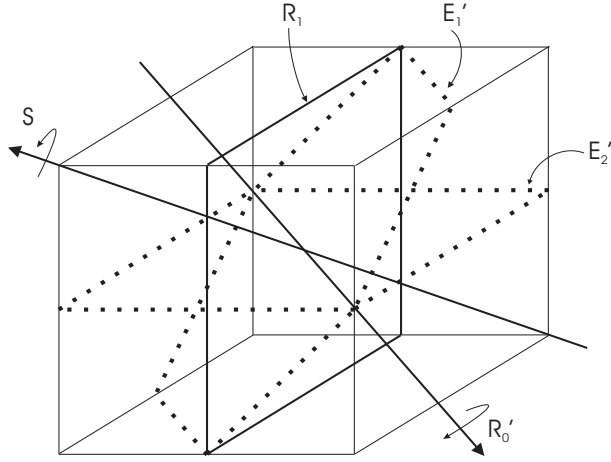


Figure 3: The special group of the complex $\mathcal{K}_2(1, 2)$

pairs of opposites, located at the centers and at common neighboring vertices of adjacent cubes. In this way, each pair of adjacent cubes determines exactly two faces of $\mathcal{K}_2(1, 2)$, and these meet only at their two vertices located at the centers of the cubes. Moreover, the three cubes that share a common vertex with a given cube and are adjacent to it, determine the three faces of $\mathcal{K}_2(1, 2)$ that surround the edge that joins the common vertex to the center of the given cube. In particular, the vertices of F_2 are o , $v = (a, -a, a)$, $(2a, 0, 0)$, (a, a, a) , in this order. Now the vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_2(1, 2)$ at o is the group $[3, 3]^*$ of order 24. The vertex-figure at o itself is the (simple) edge graph of the cube with vertices $(\pm a, \pm a, \pm a)$.

It is interesting to observe the following nice picture of the face of $\mathcal{K}_2(1, 2)$. The fundamental tetrahedron of the Coxeter group P_4 whose diagram is an unmarked square has two opposite edges of length 2 and four of length $\sqrt{3}$ (when defined relative to $2\mathbb{Z}^3$). Up to scaling, these four others give the shape of the face of $\mathcal{K}_2(1, 2)$. The same remark applies to the next complex, $\mathcal{K}_3(1, 2)$.

Finally, let G_2 be dihedral. As generators of G_2 we take the reflections \widehat{R}_2 and \widetilde{R}_2 in the planes $y = -x$ and $z = -y$, respectively, so that $\widehat{R}_2\widetilde{R}_2 = S$ with S as in (9). Then G has generators $R_0, R_1, \widehat{R}_2, \widetilde{R}_2$ given by

$$\begin{aligned}
 R_0: \quad (x, y, z) &\mapsto (y, x, -z) + (a, -a, a), \\
 R_1: \quad (x, y, z) &\mapsto (x, -y, z), \\
 \widehat{R}_2: \quad (x, y, z) &\mapsto (-y, -x, z), \\
 \widetilde{R}_2: \quad (x, y, z) &\mapsto (x, -z, -y),
 \end{aligned} \tag{10}$$

again with $a \neq 0$ (see Figure 4). They also determine a regular complex, denoted $\mathcal{K}_3(1, 2)$, with skew square faces, now six around each edge. This contains $\mathcal{K}_2(1, 2)$ as a subcomplex.

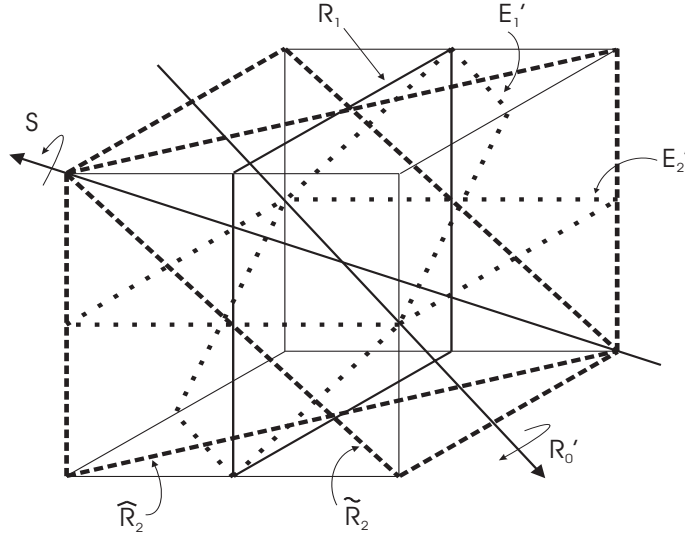


Figure 4: The special group of the complex $\mathcal{K}_3(1, 2)$

As for the previous complex, the vertex-set of $\mathcal{K}_3(1, 2)$ is $\Lambda_{(a,a,a)}$ and a typical square face has its four vertices, in pairs of opposites, located at the centers and at common neighboring vertices of adjacent cubes in the cubical tessellation with vertex set $(a, a, a) + 2a\mathbb{Z}^3$. Now each pair of adjacent cubes determines four faces of $\mathcal{K}_3(1, 2)$, not only two as for $\mathcal{K}_2(1, 2)$. The three cubes that share a vertex with a given cube and are adjacent to it, now determine all six faces of $\mathcal{K}_3(1, 2)$ that surround the edge that joins the common vertex to the center of the given cube. The base face F_2 is the same as for $\mathcal{K}_2(1, 2)$. However, the vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_3(1, 2)$ at o now is the full group $[3, 4]$, so the vertex-figure at o itself is the *double-edge graph* of the cube with vertices $(\pm a, \pm a, \pm a)$, meaning the graph obtained from the ordinary edge graph by doubling the edges but maintaining the vertices. (Note that the double-edge graph admits an action of the vertex-figure group.)

Observe also that the faces of $\mathcal{K}_3(1, 2)$ are those of the Petrie duals of the facets $\{\infty, 4\}_4 \# \{ \}$ of the 4-apeirotope $\text{apeir}\{4, 3\}$, with $\{4, 3\}$ properly chosen.

Case II: Hexagonal faces

Suppose $R'_0 R_1$ is a rotatory reflection of period 6 whose invariant plane E'_2 is perpendicular to a main diagonal of C , the diagonal passing through the vertices $\pm(1, 1, 1)$ (say). Then we may take R'_0 to be the half-turn about the line through o and $(1, 0, -1)$ (say), the latter being the midpoint of an edge of C . Moreover, since the mirror of R_1 is perpendicular to E'_2 and $R'_0 R_1$ has period 6, we may assume R_1 to be the reflection in the plane $y = x$. With R'_0, R_1 specified, we now have three choices for E'_1 , namely E'_1 is a coordinate plane or is perpendicular to either a main diagonal of C or to a line through the midpoints of a pair of antipodal edges of C .

Case IIa: E'_1 is a coordinate plane

Suppose E'_1 is the xz -plane (say), so that S rotates about the y -axis. Then G_2 is cyclic of order 2 or 4, or dihedral of order 4 or 8. In either case, the twin vertex is of the form $v = (0, a, 0)$ with $a \neq 0$.

If G_2 is cyclic of order 2, then \mathcal{K} is a polyhedron with complete dimension vector $(1, 2, 1)$. In fact, comparison with [21, p.225] shows that $K = \{6, 4\}_6$. The faces of \mathcal{K} are skew hexagons given by Petrie polygons of half the cubes in a cubical tessellation, and the vertex-figures are planar squares.

Next we eliminate the possibility that G_2 is cyclic of order 4. In fact, the presence of S as a symmetry of \mathcal{K} already forces G_2 to be dihedral; that is, G_2 cannot be cyclic. In the present configuration, S and R_1 already generate the maximum possible group, namely the full octahedral group $[3, 4] = G_*$ (see Lemma 3.8). Since the vertex-figure group $\langle R_1, G_2 \rangle$ of \mathcal{K} at o must be a subgroup of G_* containing S , it must coincide with G_* and hence contain the full dihedral subgroup D_4 of G_* that contains S . Thus D_4 is a subgroup of G . Moreover, since D_4 consists of symmetries of \mathcal{K} that fix o and v , it lies in the pointwise stabilizer G_2 of F_1 . Hence $G_2 = D_4$.

Now suppose that G_2 is dihedral of order 4. Then, in case (A), G_2 is generated by the reflections R_2 and \widehat{R}_2 in the planes $z = -x$ and $z = x$, respectively. Hence the generators $R_0, R_1, R_2, \widehat{R}_2$ of G are given by

$$\begin{aligned} R_0: (x, y, z) &\mapsto (-z, -y, -x) + (0, a, 0), \\ R_1: (x, y, z) &\mapsto (y, x, z), \\ R_2: (x, y, z) &\mapsto (-z, y, -x), \\ \widehat{R}_2: (x, y, z) &\mapsto (z, y, x), \end{aligned} \tag{11}$$

with $a \neq 0$ (see Figure 5). In particular, they yield a regular complex, denoted $\mathcal{K}_4(1, 2)$, which has skew hexagonal faces, four surrounding each edge.

The vertex-set of $\mathcal{K}_4(1, 2)$ is $a\mathbb{Z}^3$. The faces of $\mathcal{K}_4(1, 2)$ are Petrie polygons of half the cubes in the cubical tessellation of \mathbb{E}^3 with vertex set $a\mathbb{Z}^3$. In each cube occupied, the Petrie polygons of this cube determine four faces of $\mathcal{K}_4(1, 2)$, forming a finite subcomplex $\{6, 3\}_4$ (the Petrie dual of $\{4, 3\}$), such that every edge lies in exactly two of them. Thus the edge graph of $\mathcal{K}_4(1, 2)$ is the full edge graph of the cubical tessellation and every edge is surrounded by four faces, such that occupied cubes with a common edge (they are necessarily non-adjacent) contribute exactly two faces to this edge of $\mathcal{K}_4(1, 2)$. In particular, the vertices of the base face F_2 are $o, v = (0, a, 0), (0, a, -a), (a, a, -a), (a, 0, -a), (a, 0, 0)$, in this order. Now the vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_4(1, 2)$ at o is $[3, 3]$, occurring here with standard generators R_2, R_1, \widehat{R}_2 . The vertex-figure at o itself is the (simple) edge graph of the octahedron with vertices $(\pm a, 0, 0), (0, \pm a, 0), (0, 0, \pm a)$.

Let G_2 be dihedral of order 4 and consider case (B). As generators of G_2 we have the reflections $\widehat{R}_2, \widetilde{R}_2$ in the planes $x = 0$ and $z = 0$, respectively. Then $R_0, R_1, \widehat{R}_2, \widetilde{R}_2$

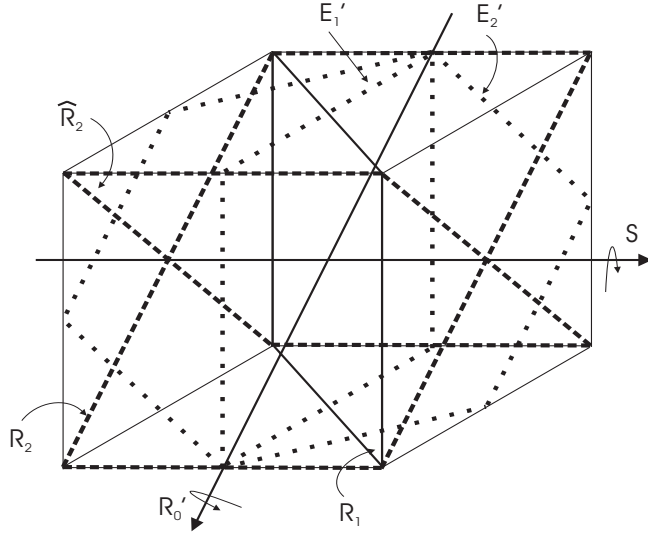


Figure 5: The special group of the complex $\mathcal{K}_4(1, 2)$

generate G and are given by

$$\begin{aligned}
 R_0: \quad (x, y, z) &\mapsto (-z, -y, -x) + (0, a, 0), \\
 R_1: \quad (x, y, z) &\mapsto (y, x, z), \\
 \widehat{R}_2: \quad (x, y, z) &\mapsto (-x, y, z), \\
 \widetilde{R}_2: \quad (x, y, z) &\mapsto (x, y, -z),
 \end{aligned} \tag{12}$$

again with $a \neq 0$ (see Figure 6). These generators give a regular complex, denoted $\mathcal{K}_5(1, 2)$, which also has skew hexagonal faces such that four surround each edge.

Now the vertex set of the complex $\mathcal{K}_5(1, 2)$ is $a\mathbb{Z}^3 \setminus ((0, 0, a) + a\Lambda_{(1,1,1)})$. As in the previous case, the faces are Petrie polygons of cubes in the cubical tessellation of \mathbb{E}^3 with vertex set $a\mathbb{Z}^3$, with F_2 exactly as before. Now each cube occurs and contributes a single face (its Petrie polygon with vertices not in $(0, 0, a) + a\Lambda_{(1,1,1)}$), and each vertex lies in exactly eight faces. In particular, the eight faces with vertex o are those Petrie polygons of the eight cubes with vertex o that have o as a vertex and have their two edges incident with o lying in the plane $z = 0$. The vertex-figure at o is the double-edge graph of the square with vertices $(\pm a, 0, 0)$ and $(0, \pm a, 0)$, meaning again the ordinary edge graph with its edges doubled. Thus the vertex-figure at o is planar, lying in the plane $z = 0$. Observe that the vertex-figure group at o is a reducible group $[4, 2] \cong D_4 \times C_2$, occurring here with generators $\widetilde{R}_2, R_1, \widehat{R}_2$ and leaving the plane $z = 0$ invariant.

It remains to consider the case that G_2 is dihedral of order 8. Then we are in case (A) and G_2 is generated by the reflections R_2 and \widehat{R}_2 in the planes $z = -x$ and $x = 0$,

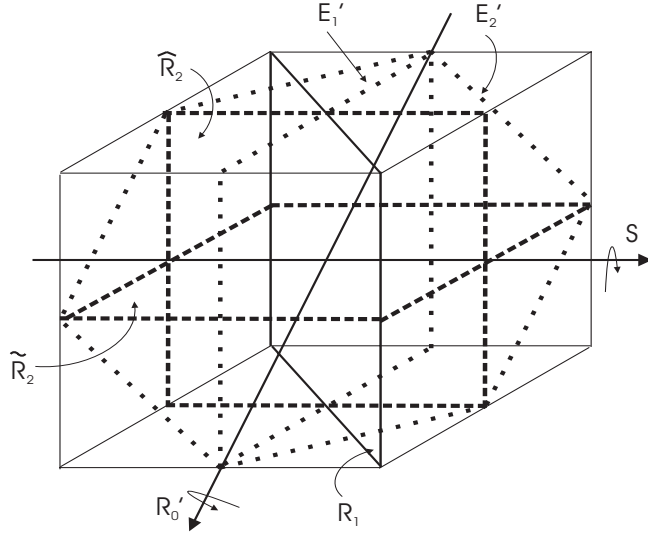


Figure 6: The special group of the complex $\mathcal{K}_5(1, 2)$

respectively. Now the generators R_0, R_1, R_2, \hat{R}_2 of G are given by

$$\begin{aligned}
 R_0: \quad (x, y, z) &\mapsto (-z, -y, -x) + (0, a, 0), \\
 R_1: \quad (x, y, z) &\mapsto (y, x, z), \\
 R_2: \quad (x, y, z) &\mapsto (-z, y, -x), \\
 \hat{R}_2: \quad (x, y, z) &\mapsto (-x, y, z),
 \end{aligned} \tag{13}$$

again with $a \neq 0$ (see Figure 7). Once more we obtain a regular complex, denoted $\mathcal{K}_6(1, 2)$, again with skew hexagonal faces but now eight surrounding each edge.

The vertex-set of $\mathcal{K}_6(1, 2)$ is the full lattice $a\mathbb{Z}^3$. As for the two previous complexes, the faces are Petrie polygons of cubes in the cubical tessellation of \mathbb{E}^3 with vertex set $a\mathbb{Z}^3$, with the base face F_2 unchanged. Here each cube occurs and contributes all four Petrie polygons, forming again a finite subcomplex $\{6, 3\}_4$. In particular, the edge graph of $\mathcal{K}_6(1, 2)$ is the full edge graph of the cubical tessellation and every edge of $\mathcal{K}_6(1, 2)$ lies in eight faces, two coming from each cube that contains it. The vertex-figure of $\mathcal{K}_6(1, 2)$ at o is the double-edge graph of the octahedron with vertices $(\pm a, 0, 0), (0, \pm a, 0), (0, 0, \pm a)$, and the vertex-figure group $\langle R_1, G_2 \rangle$ is $[3, 4]$.

Note that the faces of $\mathcal{K}_6(1, 2)$ are those of the Petrie duals of the facets $\{\infty, 3\}_6 \# \{ \}$ of the 4-apeirotope $\text{apeir}\{3, 4\}$, with $\{3, 4\}$ properly chosen (see (3)).

Case IIb: E'_1 is perpendicular to a main diagonal of C

Recall that E'_2, R'_0 and R_1 are exactly as in the previous case. Since the rotation axis of R'_0 lies in E'_1 , there is exactly one choice for E'_1 (as usual, up to congruence), namely the plane through o perpendicular to the main diagonal connecting $\pm(1, -1, 1)$. Hence G_2

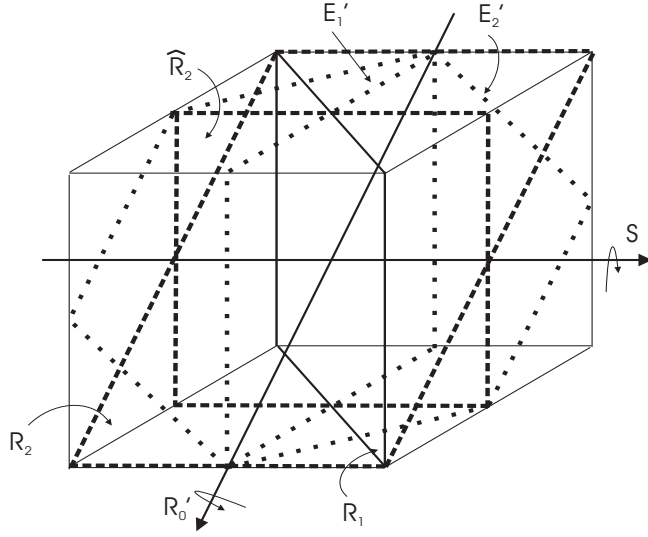


Figure 7: The special group of the complex $\mathcal{K}_6(1, 2)$

is cyclic of order 3 or dihedral of order 6, and the twin vertex is of the form $v = (a, -a, a)$ with $a \neq 0$.

We can immediately rule out the possibility that G_2 is cyclic. In fact, by Lemma 3.8, the presence of a 3-fold rotation, S , in G_2 already forces G_2 to be dihedral. Observe here that the vertex-figure group $\langle R_1, G_2 \rangle$ of \mathcal{K} at o is the subgroup $[3, 3]$ of $[3, 4]$ and hence contains the full dihedral subgroup D_3 of $[3, 3]$ that contains S ; moreover, this subgroup D_3 fixes o and v , so that $G_2 = D_3$. Thus G_2 cannot be cyclic.

Now suppose G_2 is dihedral of order 6. Then we are necessarily in case (B). As generators for G_2 we may take the reflections \hat{R}_2 and \tilde{R}_2 in the planes $y = -x$ and $z = -y$, respectively. This leads to the generators $R_0, R_1, \hat{R}_2, \tilde{R}_2$ of G given by

$$\begin{aligned}
 R_0: \quad (x, y, z) &\mapsto (-z, -y, -x) + (a, -a, a), \\
 R_1: \quad (x, y, z) &\mapsto (y, x, z), \\
 \hat{R}_2: \quad (x, y, z) &\mapsto (-y, -x, z), \\
 \tilde{R}_2: \quad (x, y, z) &\mapsto (x, -z, -y),
 \end{aligned} \tag{14}$$

with $a \neq 0$ (see Figure 8). The resulting regular complex, denoted $\mathcal{K}_7(1, 2)$, has skew hexagonal faces such that six surround each edge.

The geometry of $\mathcal{K}_7(1, 2)$ is more complicated than in the previous cases. The vertex-set is $2a\Lambda_{(1,1,0)} \cup ((a, -a, a) + 2a\Lambda_{(1,1,0)})$. The edges of $\mathcal{K}_7(1, 2)$ run along main diagonals of the cubes in the cubical tessellation with vertex-set $a\mathbb{Z}^3$, so in particular the faces are not Petrie polygons of cubes. The base face F_2 has vertices

$$\begin{aligned}
 v_0 &:= o, \quad v_1 := v = (a, -a, a), \quad v_2 := (0, -2a, 2a), \\
 v_3 &:= (-a, -a, 3a), \quad v_4 := (-2a, 0, 2a), \quad v_5 := (-a, a, a),
 \end{aligned}$$

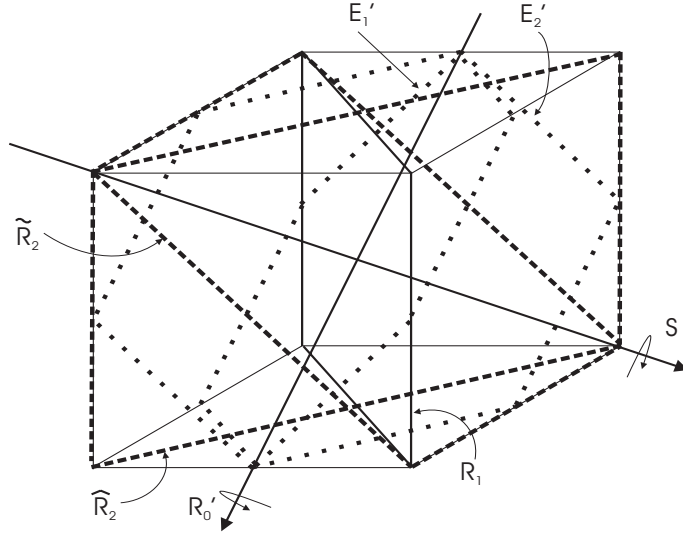


Figure 8: The special group of the complex $\mathcal{K}_7(1, 2)$

in this order. The vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_7(1, 2)$ at o is again $[3, 3]$. The vertex-figure at o itself is the double-edge graph of the tetrahedron with vertices $(a, -a, a)$, $(-a, a, a)$, $(a, a, -a)$, $(-a, -a, -a)$.

The twelve faces of $\mathcal{K}_7(1, 2)$ containing the vertex o can be visualized as follows. Consider the three (nested) cubes aC , $2aC$ and $3aC$, referred to as the *inner*, *middle* or *outer* cube, respectively; here C is our reference cube with vertices $(\pm 1, \pm 1, \pm 1)$. Recall that there are two ways of inscribing a regular tetrahedron at alternating vertices of the inner cube, one given by the tetrahedron P_a (say) with vertices $(a, -a, a)$, $(-a, a, a)$, $(a, a, -a)$, $(-a, -a, -a)$ that occurs as the vertex-figure at o . Clearly, P_a is the convex hull of the orbit of v_1 under the vertex-figure group $[3, 3]$. Let P_{2a} and P_{3a} , respectively, denote the convex hulls of the orbits of v_2 and v_3 under $[3, 3]$. Then P_{2a} is the cuboctahedron whose vertices are the midpoints of the edges of the middle cube, and P_{3a} is a truncated tetrahedron with its vertices on the boundary of the outer cube (and with four triangular and four hexagonal faces).

With these reference figures in place, a typical hexagonal face of $\mathcal{K}_7(1, 2)$ with vertex o then takes its other vertices from P_a , P_{2a} and P_{3a} , such that the vertices adjacent to o are vertices of P_a , those two steps apart from o are vertices of P_{2a} , and the vertex opposite to o is a vertex of P_{3a} . In particular, for F_2 , the vertices v_1, v_5 are from P_a , the vertices v_2, v_4 are from P_{2a} , and vertex v_3 is from P_{3a} . More precisely, since the vertex-figure is the double-edge graph of the tetrahedron, every edge e of P_a determines two faces of $\mathcal{K}_7(1, 2)$ as follows. First note that the midpoint of e lies on a coordinate axis and determines a positive or negative coordinate direction. Moving along this direction to the boundary of P_{2a} , we encounter a square face of P_{2a} with one pair of opposite edges given by translates

of e . Then each edge in this pair determines a unique face of $\mathcal{K}_7(1, 2)$ that contains o . Thus there are two faces associated with e . To find their final vertex we move even further to the boundary of P_{3a} until we hit an edge of P_{3a} . The vertices of this edge then are the opposites of o in the two faces determined by e . Note here that the orbit of an edge of P_{2a} under $[3, 3]$ consists only of twelve edges, namely those of four mutually non-intersecting triangular faces of P_{2a} . On the other hand, P_{3a} has exactly twelve vertices.

Moreover, notice that the vertex-figure at a vertex adjacent to o (such as v) similarly uses the other regular tetrahedron inscribed at alternating vertices of the inner cube. Thus both tetrahedra inscribed in the inner cube occur (in fact, already at every pair of adjacent vertices of $\mathcal{K}_7(1, 2)$). Rephrased in a different way, every vertex w of $\mathcal{K}_7(1, 2)$ lies in four edges of $\mathcal{K}_7(1, 2)$, such that their (outer) direction vectors point from w to the vertices of a tetrahedron that is the translate by w of a regular tetrahedron inscribed in the inner cube aC , and such that adjacent vertices of $\mathcal{K}_7(1, 2)$ always use different inscribed tetrahedra.

Finally, observe that the faces of $\mathcal{K}_7(1, 2)$ are those of the Petrie duals of the facets $\{\infty, 3\}_6 \# \{ \}$ of the 4-aperoitope $\text{apeir}\{3, 3\}$, with $\{3, 3\}$ properly chosen. The common edge graph of $\text{apeir}\{3, 3\}$ and $\mathcal{K}_7(1, 2)$ is the famous *diamond net* (see [21, p. 241]). The latter models the diamond crystal, with the carbon atoms sitting at the vertices and with the bonds between adjacent atoms represented by the edges (see also [32, pp. 117, 118]).

Case IIc: E'_1 is perpendicular to the line through the midpoints of a pair of antipodal edges of C

Since the rotation axis of R'_0 must lie in E'_1 , there is exactly one choice for E'_1 , namely the plane through o perpendicular to the line through $\pm(1, 0, 1)$. Hence G_2 is cyclic of order 2 or dihedral of order 4, and the twin vertex is of the form $v = (a, 0, a)$ with $a \neq 0$.

If G_2 is cyclic of order 2, then \mathcal{K} is a regular polyhedron with complete dimension vector $(1, 2, 1)$. Its faces are skew hexagons, but now the edges are parallel to face diagonals of the cube; in fact, the faces of \mathcal{K} are congruent to Petrie polygons of regular octahedra (see also $\mathcal{K}_8(1, 2)$ described below). Moreover, \mathcal{K} has planar hexagonal vertex-figures, since R_1S is a rotatory reflection of period 6 (with invariant plane perpendicular to the line through $\pm(1, 1, -1)$). It follows that \mathcal{K} must be the polyhedron $\{6, 6\}_4$ (see [21, p. 225]).

Now suppose G_2 is dihedral of order 4. Then G_2 is generated by the reflections R_2 and \widehat{R}_2 in the planes $y = 0$ and $z = x$, respectively, and we are in case (A). The generators $R_0, R_1, R_2, \widehat{R}_2$ of G are given by

$$\begin{aligned} R_0: (x, y, z) &\mapsto (-z, -y, -x) + (a, 0, a), \\ R_1: (x, y, z) &\mapsto (y, x, z), \\ R_2: (x, y, z) &\mapsto (x, -y, z), \\ \widehat{R}_2: (x, y, z) &\mapsto (z, y, x), \end{aligned} \tag{15}$$

where again $a \neq 0$ (see Figure 9). Now we obtain a regular complex, denoted $\mathcal{K}_8(1, 2)$, with skew hexagonal faces, four surrounding each edge. This contains the polyhedron $\{6, 6\}_4$ as a subcomplex.

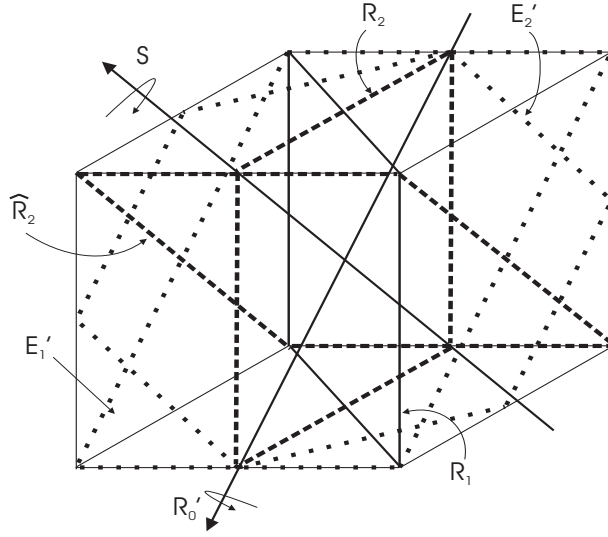


Figure 9: The special group of the complex $\mathcal{K}_8(1, 2)$

The vertex-set of $\mathcal{K}_8(1, 2)$ is $\Lambda_{(a,a,0)}$. Relative to the cubical tessellation with vertex-set $a\mathbb{Z}^3$, the edges of $\mathcal{K}_8(1, 2)$ are face diagonals and the faces of $\mathcal{K}_8(1, 2)$ are Petrie polygons of the octahedral vertex-figures of the tessellation at the vertices that are not in $\Lambda_{(a,a,0)}$. At these octahedral vertex-figures, all Petrie polygons occur as faces of $\mathcal{K}_8(1, 2)$, forming a finite subcomplex $\{6, 4\}_3$ (the Petrie dual of $\{3, 4\}$). For example, the base face F_2 has vertices $o, v = (a, 0, a), (0, -a, a), (0, 0, 2a), (-a, 0, a), (0, a, a)$, in this order, and is a Petrie polygon of the octahedral vertex-figure at $(0, 0, a)$, its center. Thus every edge of $\mathcal{K}_8(1, 2)$ is surrounded by four faces, arising in pairs from the octahedral vertex-figures at the two remaining vertices of the square face of the tessellation that has the edge as a face diagonal. Moreover, the vertex-figure group $\langle R_1, G_2 \rangle$ of $\mathcal{K}_8(1, 2)$ at o is the octahedral group $[3, 4]$, occurring here with (standard) generators R_2, R_1, \widehat{R}_2 . The vertex-figure of $\mathcal{K}_8(1, 2)$ at o is the (simple) edge graph of the cuboctahedron with vertices $(\pm a, \pm a, 0), (0, \pm a, \pm a), (\pm a, 0, \pm a)$.

Finally, then, this completes our investigation of all possible relative positions of mirrors of basic generators for the special group $G_* = [3, 4]$ under the assumption that the mirror vector is $(1, 2)$. In particular, we can summarize our results as follows.

Theorem 6.3. *Apart from polyhedra, the complexes $\mathcal{K}_1(1, 2), \dots, \mathcal{K}_8(1, 2)$ described in this section are the only simply flag-transitive regular polygonal complexes with finite faces and mirror vector $(1, 2)$, up to similarity.*

7 Complexes with infinite faces and mirror vector $(1, 2)$

In this section we complete the enumeration of the simply flag-transitive regular complexes with mirror vector $(1, 2)$. We saw in the previous section that, apart from polyhedra, there are only eight such complexes with finite faces. Now we prove that, again apart from polyhedra, there are no examples with infinite faces.

Let \mathcal{K} be a simply flag-transitive regular complex with an affinely irreducible group $G = \langle R_0, R_1, G_2 \rangle$, where again R_0, R_1, G_2 are as in Lemma 3.4, the mirror vector is $(1, 2)$, and the base vertex of \mathcal{K} is o . Then the vertex-figure group $\langle R_1, G_2 \rangle$ of \mathcal{K} at o is a subgroup of G_* , since both generators fix the base vertex, o . Recall that $v := oR_0$ is called the twin vertex of \mathcal{K} .

Now suppose \mathcal{K} has infinite faces. Then the faces must be planar zigzags. In fact, since the base face F_2 is an infinite polygon, the mirrors of the half-turn R_0 and plane reflection R_1 cannot meet and must be perpendicular to the plane E spanned by o, v and vR_1 . Hence F_2 is a zigzag contained in E , and the mirrors of R_0 and R_1 are parallel.

Now consider the elements R'_0 and R_1 of the special group G_* . Since the axis of R'_0 lies in the mirror of R_1 , their product R'_0R_1 is the reflection in the plane through o that is perpendicular to the mirror of R_1 and contains the axis of R'_0 . Note here that E is the plane through o that is perpendicular to both mirrors, of R_1 and R'_0R_1 . The twin vertex v lies in E but not on either mirror, since v is not fixed by R_1 or R'_0R_1 . Thus G_* contains a pair of reflections with perpendicular mirrors and must be a group $[3, 3]^*$, $[3, 3]$ or $[3, 4]$ (see (2)). Moreover, G_2 is cyclic or dihedral of order r . As for complexes with finite faces (and for exactly the same reasons), G_2 must contain a non-trivial rotation, S (say), generating its rotation subgroup.

Again we use the cube C with vertices $(\pm 1, \pm 1, \pm 1)$ as the reference figure for the action of G^* . As in Case I of the previous section, there are two possible choices for the half-turn R'_0 : either R'_0 rotates about the center of a face of C or about the midpoint of an edge of C . In the following we treat these as Case I and Case II. In any case, the rotation axis of R'_0 must lie in the mirrors of R_1 and R'_0R_1 .

Case I: R'_0 rotates about the center of a face of C

Suppose R'_0 rotates about the y -axis (say). Then there are two possible configurations of mirrors of R_1 and R'_0R_1 : either they are the xy -plane and yz -plane, or the planes $z = x$ and $z = -x$. In either case, E is the xz -plane and contains the rotation axis of S (passing through o and v).

The first configuration can be ruled out immediately; in fact, then S would rotate about the midpoint of an edge of C (recall that v cannot lie on either mirror) and G_* would be a reducible group with invariant plane E , regardless of whether G_2 is cyclic or dihedral. This only leaves the second configuration, for which S must also rotate about the center of a face of C .

For the second configuration we can immediately exclude the possibility that G_2 is

cyclic of order 2 or dihedral of order 4 with plane mirrors given by coordinate planes, once again appealing to irreducibility. Furthermore, G_2 can also not be cyclic of order 4 or dihedral of order 8, as can be seen as follows. First observe that, if S has period 4, then G_2 must actually be dihedral of order 8. In fact, then the vertex-figure group $\langle R_1, S \rangle$ of \mathcal{K} at o is the full group $[3, 4]$ and hence contains the full dihedral subgroup D_4 that contains S ; but this subgroup D_4 also fixes v , so that it must coincide with G_2 . On the other hand, if G_2 is dihedral of order 8, then G_2 contains the reflection with mirror E and hence E is a face mirror of \mathcal{K} ; in other words, the reflection in E fixes the base flag of \mathcal{K} , in contradiction to our assumption that \mathcal{K} is simply flag-transitive. Recall that the existence of face mirrors for G immediately forces G to be the full symmetry group of a regular 4-apeirotope in \mathbb{E}^3 (see Lemma 3.2 and Theorem 4.1). Thus we have eliminated all but one possibility for G_2 .

It remains to also reject the final possibility that G_2 is dihedral of order 4 with plane mirrors determined by face diagonals of C . In this case the configuration of mirrors and rotation axes for the generators suggests that \mathcal{K} would have to coincide with the 2-skeleton of the regular 4-apeirotope

$$\mathcal{P} := \{\{\infty, 3\}_6 \# \{\}, \{3, 4\}\} = \text{apeir}\{3, 4\}$$

(see [21, Ch.7F]). However, the 2-skeleton is not a simply flag-transitive complex, so that the final case can also be excluded once this conjecture about \mathcal{K} has been verified. We can accomplish the latter by employing Wythoff's construction as follows. Here we assume that the twin vertex is given by $v = (0, 0, a)$ with $a \neq 0$.

Let T_0, T_1, T_2, T_3 denote distinguished generators for the symmetry group $G(\mathcal{P})$ of \mathcal{P} , taken such that T_1, T_2, T_3 are distinguished generators of the octahedron $\{3, 4\}$ with vertices at the face centers of the scaled reference cube aC , and such that T_0 is the point reflection in the point $\frac{1}{2}v$ halfway between the base vertex o and twin vertex v shared by \mathcal{K} and \mathcal{P} . Here we may further assume that the generators T_1, T_2, T_3 are chosen such that $T_1 = R_1$, $S = T_2T_3$, and T_3 is the plane reflection with mirror E (the xz -plane). Then it is immediately clear that T_3 keeps the base face F_2 of \mathcal{K} pointwise fixed, since the latter lies in E . Moreover, the vertex-figure group $\langle R_1, S \rangle$ of \mathcal{K} at o is $[3, 3]$ and has index 2 in the vertex-figure group $\langle T_1, T_2, T_3 \rangle = [3, 4]$ of \mathcal{P} at o (with T_3 representing the non-trivial coset), and $T_0 = T_3R_0$. In fact, G itself is a subgroup of $G(\mathcal{P})$ of index at most 2 (with T_3 representing the non-trivial coset if the index is 2), since conjugation by T_3 leaves G invariant. Now recall that the apeirotope \mathcal{P} can be constructed by Wythoff's construction applied to $G(\mathcal{P})$ with initial vertex o (see [21, Chs.5A,7F]); in particular, its base vertex is o , its base edge is $\{o, v\}$ (the orbit of o under $\langle T_0 \rangle$), its base 2-face is the orbit of the base edge under $\langle T_0, T_1 \rangle$, and its base 3-face is the orbit of the base 2-face under $\langle T_0, T_1, T_2 \rangle$. The given complex \mathcal{K} can similarly be derived by Wythoff's construction applied to $G = G(\mathcal{K})$ with the same initial vertex, o .

We claim that \mathcal{K} coincides with the 2-skeleton of \mathcal{P} . First we verify that \mathcal{K} and the 2-skeleton of \mathcal{P} have the same base flag. Since they obviously share their base vertices

and base edges, we only need to examine their base 2-faces; but clearly these must also agree, since T_3 fixes E pointwise and $R_0 = T_3T_0$. Now recall that the flags of a regular complex are just the images of the base flag under the respective group. It follows that \mathcal{K} and the 2-skeleton of \mathcal{P} must indeed be identical, since G is a subgroup of $G(\mathcal{P})$ of index at most 2 and the only nontrivial coset of G (if any) is represented by an element, T_3 , that fixes the base flag. (It also follows at this point that G must indeed have index 2 in $G(\mathcal{P})$, since otherwise their vertex-figure subgroups would agree.)

Case II: R'_0 rotates about the midpoint of an edge of C

Suppose R'_0 is the half-turn about the line through o and $(1, 1, 0)$ (say). Then the mirrors of R_1 and R'_0R_1 are the xy -plane and the plane $y = x$, respectively, or vice versa. In either case, E is the plane $y = -x$ and S is a 3-fold rotation about a main diagonal of C contained in E .

First we deal with the case that R_1 is the reflection in the plane $y = x$. Since S is a 3-fold rotation, the vertex-figure group $\langle R_1, S \rangle$ of \mathcal{K} at o is the full tetrahedral group $[3, 3]$ and hence contains the full dihedral subgroup D_3 containing S . In particular, the reflection in E belongs to D_3 , so that E is a face mirror for \mathcal{K} ; however, this is impossible, since \mathcal{K} is simply flag-transitive (see Lemma 3.2 and Theorem 4.1).

Next suppose that R_1 is the reflection in the xy -plane. We can immediately exclude the possibility that G_2 is dihedral of order 6 (although, strictly speaking, this is subsumed under the next case). In fact, if G_2 was dihedral, the reflection in E would already lie in G_2 and hence E would again be a face mirror of \mathcal{K} .

This only leaves the possibility that G_2 is cyclic of order 3. Here the configuration of mirrors and rotation axes for the generators suggests that \mathcal{K} would have to be the 2-skeleton of the regular 4-apeirotope

$$\mathcal{P} := \{\{\infty, 4\}_4 \# \{\}, \{4, 3\}\} = \text{apeir}\{4, 3\}$$

(see [21, Ch.7F]). Hence, if confirmed, this possibility can be ruled out as well, as then \mathcal{K} could not be a simply flag-transitive complex. The conjecture that \mathcal{K} coincides with the 2-skeleton of \mathcal{P} can be verified as for the special configuration described under Case I. Let T_0, T_1, T_2, T_3 denote the distinguished generators for $G(\mathcal{P})$, where T_0 is the point reflection in the point $\frac{1}{2}v$ (with $v = (a, -a, a)$, say), and the distinguished generators T_1, T_2, T_3 of the cube $\{4, 3\}$ (taken as aC , say) are such that $T_1 = R_1$, $S = T_2T_3$, and T_3 is the plane reflection with mirror E (the plane $y = -x$). Now $\langle R_1, S \rangle$ is the subgroup $[3, 3]^*$ of index 2 in $\langle T_1, T_2, T_3 \rangle = [4, 3]$ (and does not contain T_3), and again $T_0 = T_3R_0$. Then we can argue as before. The group G is of index at most 2 in $G(\mathcal{P})$, and the reflection T_3 represents the non-trivial coset of G (if any) and fixes the common base flag of \mathcal{K} and the 2-skeleton of \mathcal{P} . Thus \mathcal{K} coincides with this 2-skeleton. (It also follows that G must have index 2 in $G(\mathcal{P})$.)

On a final note, the alert reader may be wondering why only two of the three possible 2-skeletons of regular 4-apeirotopes with infinite faces have occurred in our discussion (recall

here Lemma 4.2). In fact, the 2-skeleton of the third apeirotope $\{\{\infty, 3\}_6 \# \{ \}, \{3, 3\}\} = \text{apeir}\{3, 3\}$ has occurred as well, at least implicitly, when we rejected the first possibility under Case II that R_1 is the reflection in the plane $y = x$ (based on our observation that then G would have to be the full group of this apeirotope since E is a face mirror).

In conclusion, we have established the following theorem.

Theorem 7.1. *Apart from polyhedra, there are no simply flag-transitive regular polygonal complexes with infinite faces and mirror vector $(1, 2)$.*

The only regular polyhedra with infinite (zigzag) faces and mirror vector $(1, 2)$ are the (blended) polyhedra $\{\infty, 4\}_4 \# \{ \}$, $\{\infty, 6\}_3 \# \{ \}$ and $\{\infty, 3\}_6 \# \{ \}$, obtained by blending the Petrie duals $\{\infty, 4\}_4$, $\{\infty, 6\}_3$ and $\{\infty, 3\}_6$ of the regular plane tessellations $\{4, 4\}$, $\{3, 6\}$ and $\{6, 3\}$, respectively, with the line segment $\{ \}$ (see [21, Ch.7E]). Each polyhedron is isomorphic to the corresponding Petrie dual (its plane component) but has its two sets of alternating vertices lying in two distinct parallel planes.

Acknowledgment

We are very grateful to Peter McMullen for his comments on an earlier draft of this manuscript and for a number of very helpful suggestions. We would also like to thank an anonymous referee for a very thoughtful review with valuable suggestions for improvement.

References

- [1] J.L.Arocha, J.Bracho and L.Montejano, *Regular projective polyhedra with planar faces, Part I*, Aequat. Math. 59 (2000), 55–73.
- [2] L.Bieberbach, *Über die Bewegungsgruppen der euklidischen Räume: erste Abhandlung*, Math. Annalen 70 (1910), 297–336.
- [3] J.Bracho, *Regular projective polyhedra with planar faces, Part II*, Aequat. Math. 59 (2000), 160–176.
- [4] H.S.M.Coxeter, *Regular skew polyhedra in 3 and 4 dimensions and their topological analogues*, Proc. London Math. Soc. (2) 43 (1937), 33–62. (Reprinted with amendments in *Twelve Geometric Essays*, Southern Illinois University Press (Carbondale, 1968), 76–105.)
- [5] H.S.M.Coxeter, *Regular Polytopes* (3rd edition), Dover (New York, 1973).

- [6] H.S.M. Coxeter, *Regular Complex Polytopes* (2nd edition), Cambridge University Press (Cambridge, 1991).
- [7] L.Danzer and E.Schulte, *Reguläre Inzidenzkomplexe, I*, Geom. Dedicata 13 (1982), 295–308.
- [8] O.Delgado-Friedrichs, M.D.Foster, M.O’Keefe, D.M.Proserpio, M.M.J.Treacy and O.M.Yaghi, *What do we know about three-periodic nets?*, J. Solid State Chemistry 178 (2005), 2533–2554.
- [9] A.W.M.Dress, *A combinatorial theory of Grünbaum’s new regular polyhedra, I: Grünbaum’s new regular polyhedra and their automorphism group*, Aequationes Math. 23 (1981), 252–265.
- [10] A.W.M.Dress, *A combinatorial theory of Grünbaum’s new regular polyhedra, II: complete enumeration*, Aequationes Math. 29 (1985), 222–243.
- [11] L.C.Grove and C.T.Benson, *Finite Reflection Groups* (2nd edition), Graduate Texts in Mathematics, Springer-Verlag (New York-Heidelberg-Berlin-Tokyo, 1985).
- [12] B.Grünbaum, *Regularity of Graphs, Complexes and Designs*, in *Problèmes combinatoires et théorie des graphes*, Coll. Int. C.N.R.S. 260, Orsay (1977), 191–197.
- [13] B.Grünbaum, *Regular polyhedra — old and new*, Aequat. Math. 16 (1977), 1–20.
- [14] B.Grünbaum, *Polyhedra with hollow faces*, in *Polytopes: Abstract, Convex and Computational* (eds. T. Bisztriczky, P. McMullen, R. Schneider and A. Ivić Weiss), NATO ASI Series C 440, Kluwer (Dordrecht etc., 1994), 43–70.
- [15] B.Grünbaum, *Acoptic polyhedra*, In *Advances in Discrete and Computational Geometry*, B.Chazelle et al. (ed.), Contemp. Math. 223, American Mathematical Society (Providence, RI, 1999), 163–199.
- [16] N.W.Johnson, *Uniform Polytopes*, Cambridge University Press (Cambridge, to appear).
- [17] P.McMullen, *Regular polytopes of full rank*, Discrete & Computational Geometry 32 (2004), 1–35.
- [18] P.McMullen, *Four-dimensional regular polyhedra*, Discrete & Computational Geometry 38 (2007), 355–387.
- [19] P.McMullen, *Regular apeirotopes of dimension and rank 4*, Discrete & Computational Geometry (to appear).

- [20] P.McMullen and E.Schulte, *Regular polytopes in ordinary space*, Discrete Comput. Geom. 17 (1997), 449–478.
- [21] P.McMullen and E. Schulte, *Abstract regular polytopes*, Encyclopedia of Mathematics and its Applications, Vol. 92, Cambridge University Press, Cambridge, UK, 2002.
- [22] P.McMullen and E.Schulte, *Regular and chiral polytopes in low dimensions*, In *The Coxeter Legacy – Reflections and Projections* (eds. C.Davis and E.W.Ellers), Fields Institute Communications, Volume 48, American Mathematical Society (Providence, RI, 2006), 87–106.
- [23] B.R.Monson and A.I.Weiss, *Realizations of regular toroidal maps*, Canad. J. Math. (6) 51 (1999), 1240–1257.
- [24] M.O’Keeffe, *Three-periodic nets and tilings: regular and related infinite polyhedra*, Acta Crystallographica A 64 (2008), 425–429.
- [25] M.O’Keeffe and B.G.Hyde, *Crystal Structures; I. Patterns and Symmetry*, Mineralogical Society of America, Monograph Series, Washington, DC, 1996.
- [26] D.Pellicer and E.Schulte, *Regular polygonal complexes in space, II*, in preparation.
- [27] D.Pellicer and A.I.Weiss, *Combinatorial structure of Schulte’s chiral polyhedra*, preprint (submitted).
- [28] J.G.Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics, Springer-Verlag (New York-Berlin-Heidelberg, 1994).
- [29] E.Schulte, *Reguläre Inzidenzkomplexe, II*, Geom. Dedicata 14 (1983), 33–56.
- [30] E.Schulte, *Chiral polyhedra in ordinary space, I*, Discrete Comput. Geom. 32 (2004), 55–99.
- [31] E.Schulte, *Chiral polyhedra in ordinary space, II*, Discrete Comput. Geom. 34 (2005), 181–229.
- [32] A.F.Wells, *Three-dimensional Nets and Polyhedra*, Wiley-Interscience (New York, etc., 1977).